L-SERIES AND MODULAR FORMS OF HALF-INTEGRAL WEIGHT

RHONDA L. HATCHER

(Communicated by Dennis A. Hejhal)

Abstract. Let $f$ be a normalized Hecke eigenform of weight $2k$, with $k$ odd. The main result of this paper is an equation representing the value of $L(f, s)L(f \otimes \varepsilon, s)$ at $s = k$ in terms of the Fourier coefficients of a modular form of half-integral weight.

1. Introduction

Let $N$ be a prime, $f \in S_{2k}^{\text{new}}(I_0)$. In this paper, we will derive an equation for the value of the product of $L$-series, $L(f, k)L(f \otimes \varepsilon, k)$ at $s = k$ in terms of the squares of the Fourier coefficients of a modular form of half-integral weight. The main identity is a geometric proof of a formula due to Waldspurger [13]. The method of the proof is a generalization of the approach used by Gross [3] in proving the result for $k = 1$.

2. The Dirichlet $L$-series

Let $K$ be an imaginary quadratic field of discriminant $-D$ where $N$ is inert. Let $\mathcal{O}$ be the ring of integers in $K$ and $A$ = ideal class in $K$. Set $\varepsilon$ = quadratic character of $(\mathbb{Z}/D\mathbb{Z})^*$ determined by $\varepsilon(p) = (\frac{-D}{p})$ for $p \mid D$ and $2u = \text{number of units in } K$, and set

$$r_A(m) = \begin{cases} \text{number of ideals of norm } m \text{ in } A & \text{for } m \geq 1, \\ 1/(2u) & \text{for } m = 0. \end{cases}$$

We now define a Dirichlet series associated with $f$ and the ideal class $A$. Let $\sum_{m \geq 1} a_m q^m$, with $q = e^{2\pi i z}$, be the Fourier expansion of $f$. Define the Dirichlet $L$-series associated with $f$ and the ideal class $A$ by

$$L(f, A, s) = \sum_{(m, N) = 1} \left( \frac{\varepsilon(m)}{m^{2s-2k+1}} \right) \sum_{m \geq 1} \left( \frac{a_m r_A(m)}{m^s} \right).$$

$L(f, A, s)$ has an analytic continuation to an entire function of $s$. For proof of this fact see [4].

Received by the editors February 14, 1993.

1991 Mathematics Subject Classification. Primary 11F66; Secondary 11G40.

©1994 American Mathematical Society
0002-9939/94 $1.00 + .25$ per page

683
Let $\chi$ be a complex character of the group $\text{Pic}(\mathcal{O})$, and suppose $f = \sum_{m \geq 1} a_m q^m$ is a normalized eigenform of weight $2k$ for the Hecke algebra $T$. Define

$$L(f, \chi, s) = \sum_{A} \chi(A)L(f, A, s),$$

where the sum is over all $A$ in $\text{Pic}(\mathcal{O})$.

It can be shown that if $e(N) = 1$, then $L(f, \chi, s)$ vanishes at $s = k$. The values of $L(f, \chi, s)$ at $s = k$ in the case $e(N) = -1$ were studied in [6]. In particular, an equation representing the value of $L(f, \chi, k)$ in terms of height pairings of special points on a vector bundle $V$ is derived. In the next section we will describe the vector bundle $V$ and the equation for $L(f, \chi, k)$.

3. The representation of $L(f, \chi, k)$ in terms of a height pairing

Let $B$ be the quaternion algebra over $\mathbb{Q}$ ramified at the rational prime $N$ and at $\infty$. Let $R$ be a maximal order of $B$. Let $\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$, and let $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q} = \prod_p \mathbb{Q}_p$ be the ring of finite adèles of $\mathbb{Q}$. Set $B_p = B \otimes \mathbb{Q}_p$. Then $R_p = R \otimes \mathbb{Z}_p$ is the local component of $R$ in $B_p$. Let $\hat{B} = B \otimes \hat{\mathbb{Q}} = \prod_p B_p$ and $\hat{R} = R \otimes \hat{\mathbb{Z}} = \prod_p R_p$.

Let $B_0 = \{ b \in B : \text{Tr}(b) = 0 \}$, and let $U$ be the representation of $B^*$ on the elements $b_0 \in B_0$ with action by $B^*$ defined by $b_0\gamma = (1/N(\gamma))\gamma^{-1}b_0\gamma$. The center $Q^*$ of $B^*$ acts by $b_0x = x^{-2}b_0$ for $x \in Q^*$. An inner product on $U$ is given by $[u_1, u_2] = 1/2\text{Tr}(u_1u_2)$. Let $W_{2k-1}$ be the inner product space $\text{Sym}^{2k-2}(W)$, where $W = C_x \otimes C_y$ is the two-dimensional representation of $\text{SU}(2)$ with $[x, x] = [y, y] = 1$ and $[x, y] = 0$. Since $W_3 = \text{Sym}^2(W) \cong \{ b \in B : \text{Tr}(b) = 0 \}$ and $W_{2k-1} = \text{Sym}^{2k-2}(W) \subset \text{Sym}^{k-1}(\text{Sym}^2(W))$, we can write the inner product space $\text{Sym}^{k-1}(U)$ as the orthogonal direct sum $\text{Sym}^{k-1}(U) = W_{2k-1} \oplus M$. Note that $W_{2k-1}$ is the unique irreducible summand of highest weight $2k - 2$, and it is a representation of $B^*$ of dimension $2k - 1$.

The quaternion algebra $B$ over $\mathbb{Q}$ corresponds to an algebraic curve $Y$ of genus zero over $\mathbb{Q}$ as described in [6, p. 541]. The vector bundle $V$ is defined by

$$V = \hat{R}^* \setminus \hat{B}^* \times Y \times W_{2k-1}/B^*.$$

Let $\{I_1, I_2, \ldots, I_n\}$ be a set of left ideals of $R$ which represent the distinct ideal classes. The set $\{I_1, I_2, \ldots, I_n\}$ corresponds to a choice of elements $g_1, g_2, \ldots, g_n$ in $\hat{R}^* \setminus \hat{B}^*$ such that $\hat{B}^* = \bigcup_{i=1}^n \hat{R}^* g_i B^*$. Let $R_i$ denote the right order of $I_i$. For each $I_i = R_i/(\pm 1)$, define $W_{I_i} = \{ w \in W : \gamma(w) = w \text{ for all } \gamma \in I_i \}$. Define $\text{Pic}(V) = \bigoplus_{i=1}^n W_{I_i}$. Any $v \in V$ is equivalent to the double coset $(g_i, y, w)$ for some $i$. Define the class of the point $v$ to be the projection of $w$ in $W_{I_i}$ given by $\text{class}(v) = \frac{1}{|I_i|} \sum_{\gamma \in I_i} \gamma(w)$.

Define a mapping $\langle \ , \ \rangle$ from $V \times V$ into $\mathbb{Q}$ as follows. If $v_1 = g_1 \times y_1 \times w_1 \cong g_1 \times y_1 \times w_1$ and $v_2 = h \times y_2 \times w_2 \cong g_1 \times y_2 \times w_2$, then

$$\langle v_1, v_2 \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \sum_{\gamma \in I_i} [w_1, \gamma(w_2)] w_{2k-1} & \text{if } i = j. \end{cases}$$

This pairing induces a height pairing $\langle \ , \ \rangle$ on $\text{Pic}(V)$.
We now define a special element \( w_0 \) of \( \text{Sym}^{2k-2}(W) \). This \( w_0 \) will be used in the definition of a special point of \( V \) of discriminant \( -D \). For the remainder of the paper, we fix \( K \) an imaginary quadratic field of discriminant \( -D \), where the prime \( N \) is inert, and an embedding \( f : K \to \mathbb{B} \). Let \( w_0 = \sqrt{-D} \in \mathbb{U} \). Then \( v_0^{k-1} \) lies in \( \text{Sym}^{k-1}(U) \). Let \( w_0 \in \text{Sym}^{2k-2}(W) \) be the component of \( v_0^{k-1} \) in \( \text{Sym}^{2k-2}(W) \).

Let \( X = \tilde{R}^* \times \mathbb{B}^* \times Y/\mathbb{B}^* \). There exists a canonical identification \( Y(K) \cong \text{Hom}(K, B) \) as described in [6, p. 541]. The special points of \( V \) of discriminant \( -D \) are defined to be points of the form \( v = g \times y \times w_0 \), where \( g \times y \) lies in the image of \( \tilde{R}^* \times \mathbb{B}^* \times Y(K) \) in \( X(K) \), and the embedding \( f \) corresponding to \( y \) satisfies \( f(K) \cap g^{-1}\tilde{R} \in \mathbb{C} \), where \( \mathbb{C}_D \) is the order of discriminant \( -D \). An action of \( \text{Pic}(\mathbb{C}) \) on the special points of \( V \) of discriminant \( -D \) given by \( v \mapsto v_A \) for \( A \in \text{Pic}(\mathbb{C}) \) is defined in [6, p. 545].

Define \( e_x = \sum_A A^{-1}(A)v_A \), where the sum is over all \( A \) in \( \text{Pic}(\mathbb{C}) \). Let \( e_f, x \) be the projection of the divisor \( e_x \) in \( \text{Pic}(V) \otimes \mathbb{C} \) to the \( f \)-isotypical eigenspace for the Hecke algebra \( T \). The following result is proved in [6, Proposition 8.2, p. 558] for the case \( D \) prime, and it follows from [7, Proposition 4.1, p. 340] that the result can be extended to the case \( D \) composite.

**Proposition 1.** If \( e(N) = -1 \), then

\[
L(f, \chi, k) = \frac{(f, f)}{u^2D^{k-1}(k-1)!^2}\langle e_f, \chi, e_f, \chi \rangle,
\]

where \( \langle , \rangle \) is the Petersson product as defined in [6, p. 547].

In the special case \( \chi = 1 \), \( L(f, \chi, k) \) can be written in the form

\[
L(f, \chi, k) = L(f, k)L(f \otimes \varepsilon, k),
\]

where \( f \otimes \varepsilon = \sum_{m \geq 1} a_m \varepsilon(m)q^m \) is the twist of \( f \), \( L(f, s) = \sum_{m \geq 1} a_m m^{-s} \), and \( L(f \otimes \varepsilon, s) = \sum_{m \geq 1} a_m \varepsilon(m)m^{-s} \).

Define \( e_D \) to be the class in \( \text{Pic}(V) \) of the rational divisor

\[
\frac{1}{2u} \sum_{\text{disc}(v) = -D} (v).
\]

Then, as explained in [6, p. 560], in the special case \( \chi = 1 \) Proposition 1 becomes

**Corollary 1.** If \( e(N) = -1 \), then

\[
L(f, k)L(f \otimes \varepsilon, k) = \frac{(f, f)}{D^{k-1}(k-1)!^2}\langle e_f, D, e_f, D \rangle.
\]

In the next section we will use the result of Corollary 1 to arrive at an equation representing the value of \( L(f, k)L(f \otimes \varepsilon, k) \) in terms of the Fourier coefficients of a modular form of half-integral weight.

4. **The main identity**

Define the formal series \( g = \sum_{D > 1} e_D q^D \), with the \( e_D \) as defined above. We will need the following result.
Lemma 1. For all \( e \in \text{Pic}(V) \) the series
\[
g(e) = \sum_{D>1} (e, e_D) q^D
\]
is a modular form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4N) \) with integral coefficients.

Proof. Let \( \{\alpha_j\} \) be the basis for \( \text{Pic}(V) \) that you get by tensoring the basis \( \{e_1, e_2, \ldots, e_n\} \) of \( \text{Pic}(X) \) described in [3] with the basis of \( \text{Sym}^{2k-2}(W) \) given by \( \{x^{2k-2}, x^{2k-3}y, \ldots, y^{2k-2}\} \). Recall that \( \text{Sym}^{2k-2}(W) \) embeds into \( \text{Sym}^{k-1}(U) \) as described earlier. It will suffice to show that for every \( \alpha_j \), \( g(\alpha_j) \) is a modular form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4N) \).

Suppose \( \alpha_j = e_i \otimes x^l y^m \). Then
\[
g(\alpha_j) = g(e_i \otimes x^l y^m) = \sum_{D>1} (e_i \otimes x^l y^m, e_D).
\]

Using the fact that
\[
e_D = \sum_{i=1}^n a_i(D)(e_i \otimes w_0),
\]
where \( w_0 \) is the component of \( (\sqrt{-D})^{k-1} \) in \( \text{Sym}^{2k-2}(W) \), \( e_i^\vee = e_i/|\Gamma_i| \), and \( a_i(D) \) is the number of embeddings of \( \sigma_D \to R_i \) modulo \( R_i^* \), it follows that
\[
g(\alpha_j) = \sum_{D>1} \left( \sum_{i=1}^n a_i(D)(e_i \otimes w_0) \right) q^D = \sum_{D>1} a_i(D)[x^l y^m, w_0]q^D.
\]

Applying a result of Pizer [11, Proposition 2.10], we see that the inner product \( [x^l y^m, w_0] \) is equal to \( p(\sqrt{-D}) \), where \( p \) is a spherical polynomial of degree \( k - 1 \) in three variables \( x_1, x_2, x_3 \), with \( \sqrt{-D} = x_1\mu_1 + x_2\mu_2 + x_3\mu_3 \), where \( \{\mu_1, \mu_2, \mu_3\} \) is a basis of \( U \). Following the work of Gross in [3], we let \( S_i \) be the suborder of index \( 8 \) in \( R_i \) which is defined by \( S_i = Z + 2R_i \), and let \( S_i^0 \) be the subgroup of elements of trace zero in \( S_i \). Then \( S_i^0 \) has rank three over \( Z \). From [3], we know that \( a_i(D) \) is one half of the number of elements \( b \in R_i \) with \( b \equiv 0, 1 \mod 2R_i \), \( \text{Tr} b = 0 \), and \( Nb = D = -b^2 \). It follows that
\[
g(\alpha_j) = \sum_{D>1} a_i(D)p(\sqrt{-D})q^D = \frac{1}{2} \sum_{b \in S_i^0} p(b)q^{Nb}.
\]

Since \( Nb \) is a positive definite ternary quadratic form and \( p(b) \) is a spherical polynomial of degree \( k - 1 \) in three variables, it follows from a result of Schoeneberg [12] generalized by Pfetzer [10, p. 452] that \( g(\alpha_i) \) is a modular form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4N) \). The lemma now follows.

Define \( e_f \) to be a nonzero element in the \( f \)-isotypical component of \( \text{Pic}(V) \otimes \mathbb{R} \), where \( f \) is an eigenform for the Hecke algebra \( T \). Then from Lemma 1, we know that
\[
g(e_f) = \sum_{D} (e_f, e_D) q^D \equiv \sum_D m_D q^D
\]
is a modular form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4N) \) with integral coefficients. Furthermore, \( g(e_f) \) is an eigenform for the Hecke algebra on the space of modular...
forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4N)$ with eigenvalues corresponding to the eigenvalues of $f$. This fact follows from an argument analogous to that given by Gross in [3] for the case $k = 1$.

The main result, which we prove next, gives a representation of the value of $L(f, k)L(f \otimes \epsilon, k)$ in terms of the Fourier coefficients $m_D$. The formula in Proposition 2 is due to Waldspurger [13].

**Proposition 2.** If $\epsilon(N) = -1$, then

$$L(f, k)L(f \otimes \epsilon, k) = \frac{(f, f)}{D^{k-\frac{1}{2}}(k-1)!^2} \frac{m_D^2}{(e_f, e_f)}.$$ 

**Proof.** From Proposition 1, we know that

$$L(f, k)L(f \otimes \epsilon, k) = \frac{(f, f)}{D^{k-\frac{1}{2}}(k-1)!^2} (e_f, D, e_f, D).$$ 

Hence, it will suffice to show that

$$(e_f, D, e_f, D) = \frac{m_D^2}{(e_f, e_f)}.$$ 

To see that this is true, first note that $m_D = (e_f, e) = (e_f, e_f, D)$. Therefore,

$$e_f, D = \frac{m_D}{(e_f, e_f)} e_f$$

in $(\text{Pic}(V) \otimes \mathbb{R})^f$. It follows that

$$(e_f, D, e_f, D) = \left( \frac{m_D}{(e_f, e_f)} e_f, \frac{m_D}{(e_f, e_f)} e_f \right)$$

$$= \frac{m_D^2}{(e_f, e_f)^2} (e_f, e_f) = \frac{m_D^2}{(e_f, e_f)}.$$ 

This completes the proof.

**References**


**Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129**

*E-mail address: hatcher@gamma.is.tcu.edu*