INDECOMPOSABLE MODULES
OVER NAGATA VALUATION DOMAINS

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Abstract. For a discrete valuation ring $R$, let $fr(R)$ be the supremum of the ranks of indecomposable finite rank torsion-free $R$-modules. Then $fr(R) = 1, 2, 3,$ or $\infty$. A complete list of indecomposables is given if $fr(R) \leq 3$, in which case $R$ is known to be a Nagata valuation domain.

Let $R$ be a discrete valuation ring with prime $p$ and quotient field $Q$, and let $R^*$ be the $p$-adic completion of $R$ with quotient field $Q^*$. Define $fr(R) = \sup\{\text{rank} X : X \text{ indecomposable torsion-free } R\text{-module of finite rank}\}$. In this paper, we show that $fr(R) = 1, 2, 3,$ or $\infty$. This resolves a conjecture by P. Vamos that $fr(R) = 1, 2,$ or $\infty$.

It is well known that $fr(R) = \infty$ in case $[Q^*: Q]$ is infinite and $fr(R) = 1$ if $[Q^*: Q] = 1$. Call $R$ a Nagata valuation domain if $2 \leq [Q^*: Q]$ is finite [Z]. In this case char $Q^* = q > 0$; $Q^* = Q(u)$ for some unit $u$ of $R^*$ with $u^n = \lambda$, a unit of $R$; and $[Q^*: Q]$ is a power of $q$ [V, R]. Examples of Nagata valuation domains are given in [N] and [V].

Zanardo [Z] shows that if $[Q^*: Q] = 2$, then $fr(R) = 2$. Moreover, in this case there are, up to isomorphism, only three indecomposables: $R, Q$, and $R^*$. His example showing that $fr(R) \geq 6$ for $[Q^*: Q] = 3$ is in error.

Henceforth, assume $[Q^*: Q] = n \geq 2$. Then $Q^*$ is a splitting field for each finite rank $R$-module $X$; i.e., $R^* \otimes X$ is the direct sum of a free $R^*$-module and a $Q^*$-module. Thus, quasi-homomorphism results of Lady [L1, L3] for modules over a discrete valuation ring with a fixed splitting field are applicable.

As summarized in [L1, Theorem 1] and proved in [L3, Theorem 5.1], for:

$n = 2$, there are three strongly indecomposables: $R$, $Q$, and $R^*$;

$n = 3$, there are five strongly indecomposables: $R$, $Q$, $R^*$, $C+R$ (p-rank 1, rank 2), and $C+R^*$ (p-rank 2, rank 3);

$n = 4$, there are strongly indecomposables of arbitrarily large finite rank, but all strongly indecomposables are potentially describable (tame representation type);

$n \geq 5$, there are strongly indecomposables of arbitrarily large finite rank, but a description is generally regarded as hopeless (wild representation type).
Since strongly indecomposables are indecomposable, Lady's theorem yields $\text{fr}(R) = \infty$ for $n \geq 4$. We give an alternate proof by easily constructed examples in §3. This is sufficient for our purposes and avoids the deep arguments used in [L3].

The only unresolved case is $n = 3$. In this case, we show that $\text{fr}(R) = 3$ and give a complete list of indecomposables up to isomorphism: $R$, $Q$, $R^\ast$, $C^{-R}$, and infinitely many of $p$-rank 2, rank 3 (all quasi-isomorphic to $C^+R^\ast$). The strongly indecomposable $R$-module $C^+R^\ast$ is the quasi-homomorphism dual of $R^\ast$ defined in [A1].

1. Preliminaries

The $p$-rank of an $R$-module $X$ is the $R/pR$-dimension of $X/pX$. Fundamental properties of $p$-rank are given in [A1].

**Lemma 1.1** [A1, Proposition 1.3, Lemma 1.5]. Two finite rank $R$-modules $G$ and $H$ are quasi-isomorphic if and only if $p$-rank $G = p$-rank $H$, rank $G = \text{rank } H$, and there is a monomorphism $f: G \to H$. Moreover, quasi-isomorphism implies isomorphism for modules of $p$-rank 1.

2. Indecomposables for $[Q^*: Q] = 3$

As noted in the introduction, $\text{char } R = 3$ and $Q^* = Q(u)$ for some unit $u$ of $R^*$ with $u^3 = \lambda$, a unit of $R$. This notation is maintained throughout the rest of this section.

Define $A[u]$ to be the pure $R$-submodule of $R^*$ generated by $\{1, u\}$. Then $A[u] = (Q \oplus Qu) \cap R^*$ is strongly indecomposable with $p$-rank 1 and rank 2 and, hence, is quasi-isomorphic to $C^{-R}$ by Lady's theorem. The following lemma is proved in [Z, Proposition 5] using Kurosch matrix-invariant arguments from [A1]. However, it can also be proved directly from the definition of $A[u]$ (a proof is not included).

**Lemma 2.1** [Z, Corollary 12, Theorem 8]. The module $A[u]$ is (strongly) indecomposable. Moreover, if $X$ is an indecomposable $R$-module of rank $\leq 2$, then $X$ is isomorphic to $R$, $Q$, or $A[u]$.

Next let $a, b \in R^* \setminus R$ and define $A[a, b]$ to be the pure $R$-submodule of $R^* \oplus R^*$ generated by $(1, 0)$, $(0, 1)$, and $(a, b)$. In particular, $QA[a, b] = Q(1, 0) \oplus Q(0, 1) \oplus Q(a, b)$ and $A[a, b] = QA[a, b] \cap (R^* \oplus R^*)$. Up to isomorphism, this definition of $A[a, b]$ coincides with that of [Z]. Then $A[a, b]$ has $p$-rank 2 and rank 3. A routine argument shows that $A[a, b]$ is (strongly) indecomposable if and only if $\{1, a, b\}$ is a $Q$-independent set. In this case, $A[a, b]$ is quasi-isomorphic to $C^+R^\ast$ by Lady's theorem. Moreover, $A[u, u^2]$ is the quasi-homomorphism dual of $R^\ast$, noting that $R^\ast$ has $p$-rank 1 and rank 3.

**Lemma 2.2.** Suppose that $(a, b)$ and $(c, d)$ are $R^\ast$-vectors.

(a) If $(c, d) = s(a, b)M + P$ for an invertible $2 \times 2$ $R$-matrix $M$, a $Q$-vector $P$, and $0 \neq s \in Q$, then $A[a, b] \approx A[c, d]$.
(c) If $r$ is a unit of $R$ and $j > i$, then $A[u + pu^2, pu^2] \approx A[u, pu^2]$. 

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Proof. (a) Define an \( R^* \)-automorphism \( \phi \) of \( R^* \oplus R^* \) by \( \phi(x) = xM^{-1} \). Then \( \phi \) induces a homomorphism \( A[c, d] \to A[a, b] \) since \( (Q \oplus Q)M^{-1} \) is contained in \( Q \oplus Q \) and \( (c, d)M^{-1} = s(a, b) + PM^{-1} \). In fact, this is an isomorphism since \( A[a, b] \) and \( \phi(A[c, d]) \) are both pure rank-3 submodules of \( R^* \oplus R^* \).

(b) Let \( A = A[u, p^i u^2] \) and \( B = A[p^i u, u^2] \) with \( i \geq 1 \). There is an \( R^* \)-endomorphism \( \phi \) of \( R^* \oplus R^* \) defined by

\[
\phi(1, 0) = (1, 1) = (1, 0) + (0, 1) \in A,
\phi(0, 1) = (-u^{-2}, -p^i u^{-1} + p^{2i}) = -\lambda^{-1}(u, p^i u^2) + p^{2i}(0, 1) \in A,
\]

recalling that \( u^3 = \lambda \). Now \( \phi \) is an automorphism as

\[
\begin{pmatrix}
1 & 1 \\
-u^{-2} & -p^i u^{-1} + p^{2i}
\end{pmatrix}
\]

has determinant \( d \equiv -u^{-2} \pmod{pR^*} \), a unit of \( R^* \). Moreover, \( \phi(B) \) is contained in \( A \) since

\[
\phi(p^i u, u^2) = p^i u(1, 1) + u^2(-u^{-2}, -p^i u^{-1} + p^{2i}) = (p^i u - 1, p^{2i} u^2)
\]

\[
= p^i(u, p^i u^2) - (1, 0) \in A.
\]

It follows that \( \phi : B \to A \) is an isomorphism.

(c) Let \( A = A[u, p^i u^2] \) and \( B = A[u + p^i ru^2, p^i u^2] \), and assume that either \( i \geq 1 \) or else \( i = 0 \) and \( ru \) is not congruent to 1 modulo \( pR^* \).

Define an \( R^* \)-endomorphism \( \phi \) of \( R^* \oplus R^* \) by

\[
\phi(1, 0) = (1 - p^i ru, -p^i ru^2 + p^{2i+j} r^2 \lambda)
\]

\[
= (1, 0) - p^i r(u, p^i u^2) + p^{2i+j} r^2 \lambda(0, 1) \in A,
\phi(0, 1) = (0, 1 - p^{3i} r^3 \lambda) = (1 - p^{3i} r^3 \lambda)(0, 1) \in A.
\]

Then \( \phi \) is an automorphism if \( i \geq 1 \), since the coefficient determinant \( d = (1 - p^i ru)(1 - p^{3i} r^3 \lambda) \equiv 1 \pmod{pR^*} \). If \( i = 0 \), then \( d = (1 - ru)(1 - \lambda^3) \).

Since \( \text{char} Q^* = 3 \), \( 1 - \lambda^3 = 1 - (ru)^3 = (1 - ru)^3 \), whence \( d = (1 - ru)^4 \).

Thus, \( \phi \) is an automorphism, as \( ru \) is not congruent to 1 modulo \( pR^* \).

Now \( \phi(B) \) is contained in \( A \) since

\[
\phi(u + p^i ru^2, p^i u^2)
\]

\[
= (u + p^i ru^2)\phi(1, 0) + p^i u^2\phi(0, 1)
\]

\[
= (u + p^i ru^2)(1 - p^i ru, -p^{i+j} ru^2 + p^{2i+j} r^2 \lambda) + p^{2i} u^2(0, 1 - p^{3i} r^3 \lambda)
\]

\[
= (u - p^{2i} r^2 \lambda, p^i u^2 - p^i r^2 \lambda)
\]

\[
= (u, p^i u^2) - p^{2i} r^2 \lambda(1, 0) - p^{i+j} r^2 \lambda(0, 1) \in A,
\]

recalling that \( u^3 = \lambda \). As in the proof of (b), \( B \approx \phi(B) = A \).

It remains to show that it is sufficient to assume that either \( i \geq 1 \) or else \( i = 0 \) and \( u \) is not congruent to 1 modulo \( pR^* \). To see this, assume that \( i = 0 \) and \( ru = 1 + ps \) for some \( s = s_0 + s_1 u + s_2 u^2 \in R^* \). Then \( u + ru^2 = (2 + ps_0)u + p s_1 u^2 + ps_2 \lambda \). Since \( ps_2 \lambda \in Q \), it follows from (a) that \( B = A[u + ru^2, p^i u^2] \approx A[(2 + ps_0)u + ps_1 u^2, p^i u^2] \). As \( \text{char} Q = 3 \), \( 2 + ps_0 = -1 + ps_0 \) is a unit of \( R^* \). Thus, \( B \approx A[u + ptu^2, p^i u^2] \) for \( t = (2 + ps_0)^{-1}s_1 \) by (a). If \( i'' = p\text{-height}(pt) \geq j \), an application of (a) shows that \( B \approx A \). Otherwise, \( j < i'' \) and \( i'' \geq 1 \), as desired.
Theorem 2.3. If $X$ is an indecomposable $R$-module of rank 3, then $X$ is isomorphic to $R^*$ or $A[u, p^j u^2]$ for some $j$.

Proof. Note that $p$-rank $X \neq 0$ or 3, as $X$ is reduced with no free summands (see [A1]). If $p$-rank $X = 1$, then $X$ embeds in its completion which is isomorphic to $R^*$. Since $R^*$ also has $p$-rank 1 and rank 3, $X \cong R^*$ by Lemma 1.1.

Now assume that $X$ is indecomposable with $p$-rank 2 and rank 3. Then $X \cong A[a, b]$ with $(a, b) = (u, u^2)M$ for some $2 \times 2$ $R$-matrix $M$ with $\det M \neq 0$ [Z]. We outline another proof that avoids matrix invariants. Let $Rx \oplus Ry$ be a basic submodule of $X$ and extend to a maximal free submodule $Rx \oplus Ry \oplus Rz$ of $X$. Then $X$ embeds as a pure submodule of $R^x \oplus R^y \cong (R^x \oplus X)/d(R^x \oplus X)$, where $d(R^x \oplus X)$ is the maximal divisible submodule. It follows that $X \cong A[a, b]$, where image $z = ax \oplus by$ for $a, b \in R^*$. Since $Q^* = Q(u) = Q \oplus Qu \oplus Qu^2$, we may write $(a, b) = (u, u^2)M + P$ for some $R$-matrix $M$ and $R$-vector $P$. Apply Lemma 2.2(a) to see that, up to isomorphism, $P$ may be chosen to be 0.

In view of Lemma 2.2(a), the isomorphism class of $A[a, b]$ is preserved by invertible $R$-column operations on $M$. In particular, it suffices to assume that $M$ is of the form

$$
\begin{pmatrix}
p^k & 0 \\
p^r & p^j
\end{pmatrix}
$$

with $i < j$ and $r$ either 0 or a unit of $R$. This follows from the observation that if an element in a row has least $p$-height, then the other entry in its row can be set to 0 using an invertible $R$-column operation. Moreover, column interchange and multiplication of a column by a unit are invertible $R$-operations.

We now have $X \cong A[a, b]$ with $(a, b) = (p^k u + p^i r u^2, p^j u^2)$, $j > i$ and $r$ either 0 or a unit of $R$.

First, assume $k < i$. Then $X \cong A[u + p^{i-k} r u^2, p^{j-k} u^2]$ by Lemma 2.2(a). Moreover, $A[u + p^{i-k} r u^2, p^{j-k} u^2] \cong A[u, p^{j-k} u^2]$ via Lemma 2.2(c). Thus, $X \cong A[u, p^{j-k} u^2]$.

Now assume $k > i$. Factor out $p^i$ and apply Lemma 2.2(a) to assume, up to isomorphism, that $[a, b] = [p^{k-i} u + ru^2, p^{j-i} u^2]$. If $r = 0$, then $X \cong A[a, b] \cong A[u, p^i u^2]$ for some $t$, obtained by factoring out $p^{\min(k-i,j-i)}$ and applying Lemma 2.2(b) in the case $k-i > j-i$.

Finally assume that $r$ is a unit. Then $X \cong A[a, b] = A[ru^2 + p^{k-i} u, p^{j-i} u^2]$ \[ \cong A[u^2 + p^{i' r} u^2, p^{j'} u^2] \] for $i' = k - i$, $j' = j - i$, and $r' = r^{-1}$ (Lemma 2.2(a)). Since $(u^2)^2 = u \lambda$, substituting $v$ for $u^2$ in the latter term and relabeling exponents and units gives $X \cong A[v + p^{i'} r v^2, p^{j'} v]$ for a unit $r = r'/\lambda$ of $R$. Invertible $R$-column operations on

$$
\begin{pmatrix}
1 & p^{j'} \\
p^{i'} & 0
\end{pmatrix}
$$

reduce to the case that $X \cong A(v + p^{i'} r v^2, p^{i' r} v^2)$. However, $Q^* = Q(u) = Q(v)$ with $v^3 = \lambda^2$, a unit of $R$. Thus, Lemma 2.2, with $u$ replaced by $v$, is true. The argument of the first case then shows that $X \cong A[v, p^{i'} v u^2]$ for some $t$. Hence, by Lemma 2.2, $X \cong A[u^2, p^{t} \lambda u] \approx A[u^2, p^{t} u] \approx A[p^t u, u^2] \approx A[u, p^t u^2]$, as desired.

For finite rank torsion-free $R$-modules $G$ and $H$, define $S_G(H)$ to be the
image of the evaluation map $\text{Hom}(G, H) \otimes_R G \to H$. Fundamental properties of $SG(-)$ are given in [A2, Chapter 5] for torsion-free abelian groups of finite rank.

**Proposition 2.4.** (a) If $A[u, p^i u^2] \approx A[u, p^j u^2]$, then $i = j$.
(b) There are embeddings $A[u, p^i u^2] \to A[u, p^{i-1} u^2]$ and $A[u, p^{i-1} u^2] \to A[u, p^i u^2]$. In each case the image has index $p$.
(c) If $G$ and $H$ are indecomposable with $p$-rank 2 and rank 3, then $SG(H) = H$.

**Proof.** (a) can be proven as in [Z, Proposition 16] for the case $i = 0, j = 1$. We outline an alternate proof that avoids matrix invariants. An $R$-isomorphism $\phi: A = A[u, p^i u^2] \to B = A[u, p^j u^2]$ lifts to an $R^*$-isomorphism of completions $\phi^*: A^* = R^* \oplus R^* \to B^* = R^* \oplus R^*$. Since $\phi(u, p^i u^2) \in B$ and $\phi^{-1}(u, p^j u^2) \in A$, it follows from a computation of $p$-heights that $i = j$.

(b) There is a monomorphism $f: A[u, p^{i-1} u^2] \to A[u, p^{j-1} u^2]$ induced by an $R^*$-endomorphism $\phi$ of $R^* \oplus R^*$ with $\phi(1, 0) = (1, 0)$ and $\phi(0, 1) = (0, p)$. Moreover, there is a monomorphism $f': A[u, p^i u^2] \to A[u, p^{i-1} u^2]$ induced by $\phi'(1, 0) = (p, 0)$ and $\phi'(0, 1) = (0, 1)$. Note that $ff' = p$ and $f'f = p$. Hence, if $H_1 = A[u, p^i u^2]$, then $pH_1$ is contained in image $f$. But $p$-rank $H_1 = 2$ and $H_1$ is not isomorphic to $H_1$ by (b). It follows that $H_1/image_f \approx R/pR$. Similarly, $H_{i-1}/image_f' \approx R/pR$.

(c) For $i \geq 1$ and for $\phi'$ and $\phi$ defined as in the proof of (b), there is $g: A[p^{i-1} u, u^2] \to A[p^i u, u^2]$ induced by $\phi'$ and $g': A[p^i u, u^2] \to A[p^{i-1} u, u^2]$ induced by $\phi$ with $gg' = p$ and $g'g = p$. It now follows that if $G_i = A[p^i u, u^2]$, then $G_i/image_f \approx R/pR \approx G_{i-1}/image_f'$. In view of Theorem 2.3, it is sufficient to show that

$$f + \delta_{i-1}g, \delta_{i-1}^{-1}: H_{i-1} \oplus H_{i-1} \to H_i \quad \text{and} \quad f' + \delta_{i-1}g', \delta_{i-1}^{-1}: H_i \oplus H_i \to H_{i-1}$$

are onto, for $\delta_i$ the isomorphism $G_i = A[p^i u, u^2] \to A[u, p^i u^2] = H_i$ given in Lemma 2.2(b). Assume that $f + \delta_ig, \delta_{i-1}^{-1}$ is not onto. Since $H_i/pH_i \approx R/pR \oplus R/pR$ and $pH_i$ is properly contained in both the image of $f$ and the image of $\delta_ig, \delta_{i-1}^{-1}$, it follows that image $f = image \delta_ig, \delta_{i-1}^{-1}$. Hence, $f\delta_{i-1}(G_{i-1}) = \delta_ig(G_{i-1})$. But this is a contradiction, as can be seen by observing that $f$ is a restriction of $\phi$ and $g$ is a restriction of $\phi'$. The proof that $f' + \delta_{i-1}g'$ is onto is analogous.

**Lemma 2.5.** Assume that $X$ is a finite rank $R$-module with submodule $K$ such that $A = X/K \approx A[u, p^i u^2]$ or $A[u, p^i u^2]$ for some $i \geq 0$. If $SA(X) = X$, then $K$ is a summand of $X$.

**Proof.** It suffices to prove that $\text{End}(A[u])$ and $\text{End}(A[u, p^i u^2])$ are commutative. This is a consequence of [AR2, Theorems 5.6 and 5.8] as the abelian group proof therein carries over to modules over discrete valuation rings. Recall that $A[u]$ has $p$-rank 1 and is reduced. Hence its completion is isomorphic to $R^*$. In particular, $\text{End}(A[u])$ is isomorphic to a subring of $R^*$. Moreover, $A[u, p^i u^2]$ is quasi-isomorphic to $A[u, u^2]$ which is the dual of $R^*$, as noted above. Thus, $Q \text{End}(A[u, p^i u^2]) = Q \text{End}(A[u, u^2]) = Q \text{End}(R^*) = QR^*$. It follows that $\text{End}(A[u, p^i u^2])$ is commutative.

**Theorem 2.6.** If $X$ is a finite rank $R$-module, then $X$ is the direct sum of modules of rank $\leq 3$.
Proof. Choose pure strongly indecomposable submodules $X_i$ of $X$ with $X/(X_1 + \cdots + X_m)$ $p^k$-bounded. Each $X_j$ is isomorphic to $R$, $R^*$, $Q$, $A[u]$, or $A[u, p^i u^2]$ for some $r \geq 0$ by Lady's theorem, Lemma 2.1, and Theorem 2.3. If $X_i$ is isomorphic to the pure injective module $R^*$ or $Q$, then $X_i$ is a summand of $X$. Moreover, if $X_i \cong R$, then $X$ has a cyclic summand, since $X$ modulo the pure submodule generated by $\{X_j : j \neq i\}$ is isomorphic to $R$.

We may now assume that each $X_j$ is isomorphic to $A[u]$ of some $A[u, p^i u^2]$. By induction on rank $X$ and $|X/(X_1 + \cdots + X_m)|$, it suffices to further assume that $X/(X_1 + \cdots + X_m) \cong R/pR$ and prove that $X$ has a summand of rank $\leq 3$.

Write $X = (X_1 + \cdots + X_m) + R(x_1 + \cdots + x_m)/p$. Let $K$ be the pure submodule of $X$ generated by $\{X_j : j \neq i\}$ and $A = X/K$, quasi-isomorphic to $X_1$. Then $A$ has $p$-rank $1$, rank $2$ or $p$-rank $2$, rank $3$ and has no free summands, being quasi-isomorphic to a strongly indecomposable $X_1$. Hence, $A$ is indecomposable $[A1, Proposition 4.1]$.

It is now sufficient to prove that $S_A(X) = X$, in which case $X$ has a summand isomorphic to $A$ of rank $\leq 3$ by Lemma 2.5. There is some $Y = X_i$, say $i = 1$, with $S_Y(X_j) = X_j$ for each $j$. This follows from the natural exact sequence $A[u, p^i u^2] \to A[u] \to 0$, Proposition 2.4(c), and the assumption that each $X_j \cong A[u]$ or $A[u, p^i u^2]$. Moreover, for $A = X/K \cong X_1 + R(x_1/p)$, $S_A(X_j) = X_j$ for each $j$, again by Proposition 2.4(c) or Lemma 2.1 and the fact that $A$ is indecomposable.

Write $X_i' = pX_i + Rx_i$, an indecomposable module for the same reason that $A$ is. For each $i$, there is $y_i \in A$, a unit $r_i$ of $R$, and $f_i : A \to X_i'$ with $f_i(y_i) \equiv r_i x_i \pmod{pX_i}$. This is because if $X_i' = S_A(X_i')$ is contained in $pX_i$, then $x_i \in pX_i$ and letting $r_i = 1$ will do. Note, for future reference, that we may as well assume that $f_i(y_i) \equiv x_i \pmod{pX_i}$. To see this, choose a unit $s_i$ of $R$ with $1 = r_i s_i + p t_i$, $t_i \in R$. Then $s_i f_i(y_i) \equiv x_i \pmod{pX_i}$, as desired.

We begin with the case $m = 2$ and find $x \in A$ and $g_i : A \to X_i'$ with $g_i(x) \equiv x_1 \pmod{pX_1}$ and $g_2(x) \equiv x_2 \pmod{pX_2}$. If either $f_1(y_2) \equiv s_1 x_1 \pmod{pX_1}$ or $f_2(y_1) \equiv s_2 x_2 \pmod{pX_2}$ for units $s_1, s_2$ of $R$, then let $x = y$, respectively, $x = y_1$. Otherwise, $f_1(y_2) \in X_1$ and $f_2(y_1) \in X_2$. In this case, let $x = y_1 + y_2$. In any case, there are units $t_i$ of $R$ with $f_i(x) \equiv t_i x_i \pmod{pX_i}$. As above, choose $g_i$ to be an appropriate $R$-unit multiple of $f_i$.

Next let $A' = pA + Rx$, an indecomposable submodule of $A$ for the same reason that $A$ is indecomposable. Restriction induces a well-defined $\phi = g_1 \oplus g_2 : A' \to pX = pX_1 + pX_2 + R(x_1 + x_2)$ with $\phi(x) \in (x_1 + x_2) + pX_1 + pX_2$. Since $S_A(A') = A'$ by Proposition 2.4(c) and $S_A(X_i) = X_i$ for each $i$, it follows that $S_A(pX) = pX$ and so $S_A(X) = X$. This completes the proof for $m = 2$.

We illustrate an induction argument with $m = 3$. From the $m = 2$ case $S_A(X_{12'}) = X_{12'}$ for $X_{12'} = pX_1 + pX_2 + R(x_1 + x_2)$. Consequently, there is $x \in A$ and $g_{12} : A \to X_{12'}$ with $g_{12}(x) \equiv (x_1 + x_2) \pmod{pX_1 + pX_2}$. Otherwise, $x_1 + x_2 \in S_A(X_{12'}) = X_{12'}$ is contained in $pX_1 + pX_2$. Recall that there is $y_3 = A$ and $f_3 : A \to X_3'$ with $f_3(y_3) \equiv x_3 \pmod{pX_3}$. If $f_3(x) \equiv s_3 x_3 \pmod{pX_3}$ for some unit $s_3$ of $R$, then let $a = x$. If $f_1(y_2) \equiv s(x_1 + x_2) \pmod{pX_1 + pX_2}$ for some unit $s$ of $R$, then let $a = y_3$. Otherwise, let $a = x + y_3$. It follows that $a \in A$ with $g_{12}(a) \equiv r(x_1 + x_2) \pmod{pX_1 + pX_2}$ and $f_3(a) \equiv r_3 x_3 \pmod{pX_3}$ for units $r$ and $r_3$ of $R$. As in the $m = 2$ case, we may assume that $r = r_3 = 1$ and construct $\phi : A' = pA + Ra \to pX$ with $\phi(a) \equiv (x_1 + x_2 + x_3)$.
The proof is concluded by an induction on \( m \); the argument for passing from \( m \) to \( m + 1 \) is analogous to that of the preceding paragraph.

3. INDECOMPOSABLES FOR \([Q^*: Q] = n \geq 4\)

The following are examples showing that \( \text{fr}(R) = \infty \) for \([Q^*: Q] = n \geq 4\). The detailed computations needed to verify that the modules are actually strongly indecomposable are omitted.

**Example 3.1.** Assume \( n \geq 4 \). Given \( m \geq 2 \), there is a strongly indecomposable \( R \)-module with \( p \)-rank \( m \) and rank \( 2m \).

**Proof.** Case I: \( \text{char } Q^* \geq 5 \). Since \( Q^* \) is purely inseparable over \( Q \), there is \( u \in R^* \) such that \( 1, u, u^2, u^3, \) and \( u^4 \) are \( Q \)-independent. Let \( M \) be an \( m \times m \) simple Jordan block \( R \)-matrix, i.e., the diagonal elements of \( M \) are a fixed unit \( \lambda \) of \( R \), the super diagonal elements are all 1’s, and the remaining entries are 0. Define \( X = A[\Gamma] = (R)^m \cap (Q^m \oplus Q^m \Gamma) \), where \( \Gamma = uM + u^2 I_m \), and \( R \)-module with \( p \)-rank \( m \) and rank \( 2m \).

It can be shown that \( \text{End}(X) \) is represented by the set of \( 2m \times 2m \) \( R \)-matrices \( \left( \begin{smallmatrix} \Pi & 0 \\ 0 & \Pi \end{smallmatrix} \right) \) with \( \Pi M = M \Pi \). This can be seen by equating \( Q \)-coefficients \( 1, u, u^2, u^3, \) and \( u^4 \). Consequently, \( Q \text{End}(X) \approx Q[\Gamma] \approx Q[x]/((x - \lambda)^m) \) is a ring with no nontrivial idempotents, whence \( X \) is strongly indecomposable.

Case II: \( \text{char } Q^* = 3 \). If there is \( u \in R^* \) with \( 1, u, u^2, u^3, \) and \( u^4 \) \( Q \)-independent, the construction of Case I suffices. Otherwise, there are \( u, v \in R^* \) with \( u^3, v^3 \in R \) and \( 1, u, v, u^2v, v^2u, u^2v^2, u^2, \) and \( v^2 \) are \( Q \)-independent. Choose \( M \) as in Case I, and define \( X = A[\Gamma] \) for \( \Gamma = uM + vI \). An argument similar to that of Case I shows that \( Q \text{End}(X) \approx Q[x]/((x - \lambda)^m) \) and \( X \) is strongly indecomposable.

Case III: \( \text{char } Q^* = 2 \). We are left with two possibilities not covered in Case I: there is \( u \in R^* \) with \( 1, u, u^2, \) and \( u^3 \) \( Q \)-independent and \( u^4 \in R \), or there is \( u, v \in R^* \) with \( u^2, v^2 \in R \) and \( \{1, u, v, uv\} \) \( Q \)-independent. In the first case, define \( X = A[\Gamma] \) for \( \Gamma = uM + u^3 I \). An argument similar to that of Case I shows that \( X \) is strongly indecomposable.

For the second case, define \( X = A[\Gamma] \) for \( \Gamma = uM + vI \). Once again, it can be shown that \( Q \text{End}(X) \) has no nontrivial idempotents, but the argument is slightly more complicated than the previous cases.

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**References**


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