PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 3, November 1994

# SPACES IN WHICH THE NONDEGENERATE CONNECTED SETS ARE THE COFINITE SETS

GARY GRUENHAGE

(Communicated by Franklin D. Tall)

ABSTRACT. Assuming the continuum hypothesis (CH), we construct a perfectly normal space $X$ such that $Y \subset X$ is connected and nondegenerate iff $X \backslash Y$ is finite. We also show that completely regular, as well as countable Hausdorff, examples of this kind can be obtained under axioms weaker than CH, e.g., Martin's Axiom.

## 0. INTRODUCTION

Watson [W] has asked if there are ($T_1$-) spaces satisfying the condition of the title. The following result answers this question in the affirmative, assuming the continuum hypothesis (CH) or Martin's Axiom (MA) (recall that CH $\Rightarrow$ MA).

**Main Theorem.** *Under MA, there are completely regular, as well as countable Hausdorff, spaces in which the nondegenerate connected sets are precisely the cofinite sets. Under CH, there is a perfectly normal example of this kind.*

In §1.1 we present a lemma which is a key to all of the examples. In §§2 and 3 we will present the countable Hausdorff and perfectly normal examples using CH. In §4 we show how one can modify the CH constructions to obtain the promised MA examples and discuss other variations (e.g., getting spaces in which the connected sets are the co-countable sets).

The following notation is used throughout. If $W$ is a set, $Fn(W, 2)$ is the set of all partial functions $\sigma: W \to 2$ with finite domain. If $\lambda$ is a cardinal, $[W]^\lambda$ (resp., $[W]^{<\lambda}$) denotes the set of all subsets of $W$ of cardinality $\lambda$ (resp., less than $\lambda$) and $A \subset^*_\lambda B$ means $|A \backslash B| < \lambda$; if $\lambda = \omega$, we omit the subscript, i.e., $A \subset^* B$ means $A \backslash B$ is finite. Finally, $A$ is a co-$< \lambda$ subset of $X$ if $|X \backslash A| < \lambda$.

Before proceeding with the examples, we prove a simple theorem about the general structure of these spaces. The theorem includes a result essentially due to A. H. Stone and presented in [E] that such spaces cannot contain convergent sequences; in particular, there are no examples which are metrizable, or first-countable, etc.

Received by the editors June 19, 1991 and, in revised form, February 15, 1993.
1991 *Mathematics Subject Classification*. Primary 54A35, 54D05.

**Theorem 0.1.** *Let $X$ be a space in which the nondegenerate connected sets are precisely the cofinite sets. Then*:

(a) *$X$ is a $T_1$-space; and*
(b) *every infinite subset of $X$ contains an infinite closed discrete set (in particular, $X$ cannot contain a convergent sequence).*

*Proof*. (a) If $q \neq p$ and $q \in \overline{\{p\}}$, then $\{p, q\}$ is a co-infinite connected set. Thus every point of $X$ is closed.

(b) First we establish that every $Y \in [X]^\omega$ must contain an infinite subset $Z$ such that $Z^0 = \varnothing$. Suppose not. Let $\mathscr{A}$ be an uncountable almost-disjoint family of infinite subsets of $Y$. Since $A^0 \neq \varnothing$ for each $A \in \mathscr{A}$ and $Y$ is countable, there are $A, B \in \mathscr{A}$ with $A^0 \cap B^0 \neq \varnothing$. Then $A^0 \cap B^0$ is open and finite, hence closed. Thus $X$ is not connected, a contradiction.

Now we show that every $Y \in [X]^\omega$ contains an infinite closed set. By the previous paragraph, we may assume $Y^0 = \varnothing$, i.e., $X \backslash Y$ is dense in $X$. Let $U$, $V$ be a relatively clopen partition of $X \backslash Y$, and let $U^*$, $V^*$ be open in $X$ with $U^* \cap (X \backslash Y) = U$ and $V^* \cap (X \backslash Y) = V$. Since $X \backslash Y$ is dense, $U^* \cap V^* = \varnothing$. Thus $U^* \cup V^*$ is not connected, so $H = X \backslash (U^* \cup V^*)$ is an infinite closed subset of $Y$.

Finally we show that every $Y \in [X]^\omega$ contains an infinite closed discrete set. By the previous paragraph, we may assume $Y$ is closed. Suppose $Y$ does not contain an infinite closed discrete set. Then $Y$ is countably compact. Let $Y = \{y_n\}_{n<\omega}$. By the previous paragraph, one can inductively construct a descending sequence $\{H_n\}_{n<\omega}$ of infinite closed sets with $y_n \notin H_n$. Then $\bigcap_{n=1}^\infty H_n = \varnothing$, a contradiction. □

## 1. Key lemma

The following lemma is basic to all our examples; we apply it with $\lambda = \omega$ to get the examples mentioned in Main Theorem.

**Lemma 1.1.** *Let $X$ be a set, $\kappa$ an ordinal, $\lambda$ a cardinal, and $\{A_\alpha\}_{\alpha<\kappa} \subset [X]^\lambda$ with $\bigcap_{\alpha<\kappa} A_\alpha = \varnothing$. Suppose that for each $\alpha < \kappa$, $\{U_{\alpha 0}, U_{\alpha 1}\}$ is a partition of $X \backslash A_\alpha$ and that these partitions satisfy*:

(a) *$\forall \sigma \in Fn(\kappa, 2)$, $U(\sigma) \overset{\text{def}}{=} \bigcap_{\alpha \in \operatorname{dom} \sigma} U_{\alpha\sigma(\alpha)}$ has cardinality $\geq \lambda$; and*
(b) *if $\beta < \alpha < \kappa$, $A \subset A_\beta$ is a finite Boolean combination of $\{A_\beta\} \cup \{U_{\eta e} : \beta \leq \eta < \alpha,\ e < 2\}$, and $|A \backslash A_\alpha| = \lambda$, then $|U_{\alpha e} \cap A| = \lambda$ for each $e < 2$.*

*Then $\{U(\sigma) : \sigma \in Fn(\kappa, 2)\}$ is a base for a topology $\Sigma$ on $X$ in which every* co-$< \lambda$ *subset is connected.*

*Proof*. Assume the hypotheses. First note that if $U(\sigma) \cap U(\tau) \neq \varnothing$, then $\sigma$ and $\tau$ are compatible and $U(\sigma) \cap U(\tau) = U(\sigma \cup \tau)$; it follows that the $U(\sigma)$'s form a base for a topology $\Sigma$. Now we must show that co-$< \lambda$ subsets of $X$ are $\Sigma$-*connected*. We begin by establishing a couple of facts.

**Fact 1.2.** *If $A_\beta \subset_\lambda^* \bigcup_{i<n} U_{\alpha_i \varepsilon_i}$, where each $\alpha_i \geq \beta$, then for some $i \neq j < n$, $\alpha_i = \alpha_j > \beta$ and $e_i \neq e_j$.*

*Proof of Fact* 1.2. Proof is by induction on $n$. Note that, without loss of generality, $\alpha_i \neq \beta$ for each $i < n$.

Suppose $A_\beta \subset^*_\lambda U_{\alpha_0 e_0}$. Since $A_{\alpha_0} \cap U_{\alpha_0 e_0} = \varnothing$ for $e < 2$, $|A_\beta \backslash A_{\alpha_0}| = \lambda$. By (b), $|U_{\alpha_0, 1-e_0} \cap A_\beta| = \lambda$, a contradiction. Thus Fact 1.2 holds for $n = 1$.

Assume Fact 1.2 for $n-1$, and suppose $A_\beta \subset^*_\lambda \bigcup_{i<n} U_{\alpha_i e_i}$. We can assume $A_\beta \not\subset^*_\lambda \bigcup_{i<n-1} U_{\alpha_i e_i}$. Hence if $A = A_\beta \backslash \bigcup_{i<n-1} U_{\alpha_i e_i}$, then $|A \backslash A_{\alpha_{n-1}}| = \lambda$ (else $U_{\alpha_{n-1} e_{n-1}}$ does not almost contain $A \bmod \lambda$), so $|A \cap U_{\alpha_{n-1}, 1-e_{n-1}}| = \lambda$ by (b), whence $A_\beta \not\subset^*_\lambda \bigcup_{i<n} U_{\{\alpha_i e_i\}}$, a contradiction.

**Fact 1.3.** *Let $F \in [X]^{<\lambda}$ and $O$ a clopen set in $X \backslash F$ (in the subspace topology generated by $\Sigma$). Let $\sigma \in Fn(\kappa, 2)$. If $U(\sigma) \cap (X \backslash F) \subset O$, then $O^c \overset{\text{def}}{=} (X \backslash F) \backslash O \subset \bigcup_{\alpha \in \operatorname{dom} \sigma} U_{\alpha, 1-\sigma(\alpha)}$.*

*Proof of Fact* 1.3. Let $x \in O^c$. Then for some $\tau \in Fn(\kappa, 2)$, $x \in U(\tau) \cap (X \backslash F) \subset O^c$. Hence $U(\sigma) \cap U(\tau) \subset F$. By (a), $\sigma$ and $\tau$ must be incompatible, i.e., $\sigma(\alpha) \neq \tau(\alpha)$ for some $\alpha \in \operatorname{dom} \sigma \cap \operatorname{dom} \tau$. Thus $x \in U_{\alpha\tau(\alpha)} = U_{\alpha, 1-\sigma(\alpha)}$.

Now we complete the proof of the lemma. Let $F \in [X]^{<\lambda}$, and suppose $X \backslash F$ is disconnected. Then there is a nonempty proper clopen subset $O$ of $X \backslash F$. Call $\sigma \in Fn(\kappa, 2)$ *minimal* if

$$U(\sigma) \cap (X \backslash F) \subset O \quad (\text{resp., } O^c)$$

but

$$U(\sigma') \cap (X \backslash F) \not\subset O \quad (\text{resp., } O^c)$$

for any $\sigma' \subsetneq \sigma$.

Then $O$ and $O^c$ are unions of sets of the form $U(\sigma) \backslash F$ with $\sigma$ minimal. Let $\beta$ be the least ordinal appearing in the domain of such a $\sigma$, say $\sigma_0$. Without loss of generality, $U(\sigma_0) \cap (X \backslash F) \subset O$. Suppose $U(\sigma_1) \cap (X \backslash F) \subset O^c$, $\sigma_1$ minimal. By Fact 1.3,

$$X \backslash F = O \cup O^c \subset \bigcup \{U_{\alpha, 1-\sigma_i(\alpha)} : i < 2,\ \alpha \in \operatorname{dom} \sigma_i\},$$

so

$$A_\beta \subset^*_\lambda \bigcup \{U_{\alpha, 1-\sigma_i(\alpha)} : i < 2,\ \alpha \in \operatorname{dom} \sigma_i\}.$$

It follows from Fact 1.2 that $\sigma_0(\alpha) \neq \sigma_1(\alpha)$ for some $\alpha \in \operatorname{dom} \sigma_0 \cap \operatorname{dom} \sigma_1$, $\alpha > \beta$. Hence $U(\sigma_1) \subset U_{\alpha, 1-\sigma_0(\alpha)}$. As $\sigma_1$ was arbitrary, we have

$$O^c \subset \bigcup \{U_{\alpha, 1-\sigma_0(\alpha)} : \alpha \in \operatorname{dom} \sigma_0,\ \alpha > \beta\}.$$

Hence $X \backslash O^c = O \cup F \supset \bigcap \{U_{\alpha\sigma_0(\alpha)} : \alpha \in \operatorname{dom} \sigma_0,\ \alpha \neq \beta\}$. Let $\sigma = \sigma_0 \restriction [(\operatorname{dom} \sigma_0) \backslash \{\beta\}]$. Then $U(\sigma) \cap (X \backslash F) \subset O$, contradicting the minimality of $\sigma_0$. □

## 2. Countable Hausdorff example

**Example 2.1** (CH). A countably infinite Hausdorff space in which the nondegenerate connected sets are precisely the cofinite sets.

*Proof.* Let $Q$ denote the rational numbers, and let $\{(A_\alpha, \{p_\alpha, q_\alpha\})\}_{\alpha<\omega_1}$ index all pairs $(A, \{p, q\})$ such that:

(i) $A \in [Q]^\omega$ and has at most one limit point; and
(ii) $p, q \in Q \backslash A$, $p \neq q$.

We construct $\{U_{\beta 0}, U_{\beta 1}\}_{\beta<\omega_1}$ such that, for each $\alpha < \omega_1$:

(a) $\{U_{\alpha 0}, U_{\alpha 1}\}$ is a partition of $Q \backslash A_\alpha$;
(b) $|U_{\alpha e} \cap \{p_\alpha, q_\alpha\}| = 1$ for each $e < 2$;
(c) $\forall \sigma \in Fn(\alpha+1, 2)$, $U(\sigma) \stackrel{\text{def}}{=} \bigcap_{\beta \in \operatorname{dom} \sigma} U_{\beta\sigma(\beta)}$ is dense in $Q$; and
(d) if $\beta < \alpha$ and $A \subset A_\beta$ is a finite Boolean combination of $\{A_\beta\} \cup \{U_{\eta e}\}_{\beta \le \eta < \alpha,\ e<2}$ and $|A \backslash A_\alpha| = \omega$, then for each $e < 2$, $|U_{\alpha e} \cap A| = \omega$.

To start, let $\{U_{00}, U_{01}\}$ be any partition of $Q \backslash A_0$ satisfying (b), such that $U_{0e}$ is dense in $Q$ for each $e < 2$.

Suppose $U_{\beta e}$ has been defined for all $\beta < \alpha$ and $e < 2$ and satisfying (a)–(d). Let $\{X_n\}_{n<\omega}$ index all $U(\sigma)$'s for $\sigma \in Fn(\alpha, 2)$ and all sets $A$ satisfying the hypotheses of (d), with each set listed infinitely often. Let $\{B_n\}_{n<\omega}$ enumerate a countable base for $Q$. Choose

$$x_n \neq y_n \in X_n \backslash (A_\alpha \cup \{x_i, y_i\}_{i<n} \cup \{p_\alpha, q_\alpha\})$$

such that if $X_n$ is the $k$th appearance of $U(\sigma)$, then $x_n, y_n \in B_k$. Let $U_{\alpha 0} = \{p_\alpha\} \cup \{x_n\}_{n<\omega}$, $U_{\alpha 1} = Q \backslash (U_{\alpha 0} \cup A_\alpha)$. This completes the inductive construction; it is routine to check that (a)–(d) are satisfied.

Let $\Sigma$ be the topology on $Q$ generated by the $U(\sigma)$'s. Clearly $\Sigma$ is Hausdorff. By Lemma 1.1, cofinite subsets of $Q$ are connected.

Suppose $X \subset Q$ is infinite. We will show that $Q \backslash X$ is totally disconnected. Let $p, q \in Q \backslash X$, $p \neq q$. There is $A \in [X]^\omega$ such that $A$ has at most one (Euclidean) limit point in $Q$. Then $(A, \{p, q\}) = (A_\alpha, \{p_\alpha, q_\alpha\})$ for some $\alpha < \omega_1$, and $\{U_{\alpha e} \cap (Q \backslash X)\}_{e<2}$ is a clopen partition of $Q \backslash X$ splitting $p$ and $q$. Thus $Q \backslash X$ is totally disconnected. □

We point out by way of comparison that Tzannes [Tz] has constructed in ZFC countable Hausdorff spaces in which the nondegenerate connected sets form a filter (but not the cofinite filter), answering a question of Tsvid [Ts].

## 3. A PERFECTLY NORMAL EXAMPLE

Before giving the construction, we describe the idea for at least a regular example. The set for the space will be the plane $R^2$. Suppose $\Sigma$ is a topology generated by partitions of the complements of certain countable subsets $A_\alpha$ of $R^2$ and satisfying the conditions of Lemma 1.1 with $\lambda = \omega$. While $\Sigma$ is not itself regular, it is not difficult to check that if each $A_\alpha$ is closed discrete in the Euclidean topology $\mathscr{E}$, then the topology $\rho$ generated by $\Sigma \cup \mathscr{E}$ is regular. Unfortunately, $\rho$ will not automatically have the property that cofinite sets are connected. But it can be made to have this property: we make the $U(\sigma)$'s "large" enough (they will at least be Bernstein sets in $R^2$; see (5) of Lemma 3.1 for the precise condition) so that if $U(\sigma) \cap E$, $E \in \mathscr{E}$, is contained in a $\rho$-clopen subset $O$ of a cofinite subset of $R^2$, then $U(\sigma) \subset O$. Thus any $\rho$-clopen partition of a cofinite set is a $\Sigma$-clopen partition, which is impossible by Lemma 1.1. Perfect normality is also obtained by making the $U(\sigma)$'s large enough (condition (4) of Lemma 3.1, essentially).

The remaining problem with the above is that not every infinite subset of $R^2$ contains an infinite closed discrete set, so if only $\mathscr{E}$-closed discrete sets are used for the $A_\alpha$'s one cannot show that coinfinite sets are disconnected. The way around this is to use instead of $\mathscr{E}$ a finer topology $\mathscr{T}$ on $R^2$ which has the

property that every infinite set contains an infinite $\mathscr{T}$-closed discrete set and which approximates $\mathscr{E}$ well enough to get everything in the previous paragraph to go through.

It is time for the details. The construction of the partitions is accomplished by the following lemma.

**Lemma 3.1** (CH). *Let $\{(A_\alpha, \{p_\alpha, q_\alpha\})\}_{\alpha<\omega_1}$ index pairs $(A, \{p, q\})$ such that $A \in [R^2]^\omega$ and has at most one Euclidean limit point and $p \neq q \in R^2 \backslash A$. Then there exists $\{(U_{\alpha 0}, U_{\alpha 1})\}_{\alpha<\omega_1}$ such that:*

(1) *$\{U_{\alpha 0}, U_{\alpha 1}\}$ is a partition of $R^2 \backslash A_\alpha$ which separates $\{p_\alpha, q_\alpha\}$;*

(2) *for each $\alpha < \omega_1$ and $\sigma: \alpha \to 2$, $U(\sigma) = \bigcap_{\beta \in \operatorname{dom} \sigma} U_{\alpha\sigma(\beta)}$ is Bernstein in $R^2$ (i.e., it and its complement meet every uncountable closed subset of $R^2$);*

(3) *if $\beta < \alpha$ and $A \subset A_\beta$ is a finite Boolean combination of $\{A_\beta\} \cup \{U_{\mu e} : \beta \leq \mu < \alpha,\ e < 2\}$ and $|A \backslash A_\alpha| = \omega$, then for each $e < 2$, $|U_{\alpha e} \cap A| = \omega$;*

(4) *if $C \subset R^2$ has uncountable closure, then there exists $\alpha < \omega_1$ such that for each $\sigma \in Fn(\omega_1 \backslash \alpha, 2)$,*

$$\mathrm{cl}_{\mathscr{E}}(U(\sigma) \cap C) \supset \{x : x \text{ is a condensation point of } \mathrm{cl}_{\mathscr{E}}(C)\};$$

(5) *suppose $(\tau, T, \theta)$ satisfies:*

(i) *$\tau \in Fn(\omega_1, 2)$;*

(ii) *$T$ is a tree of height $\omega$ of basic open subsets of $R^2$ (under reverse inclusion, from a fixed countable base) such that if $b$ is a branch of $T$, then $\bigcap b$ is a single point (which we also denote by $\bigcap b$); furthermore, the set $\{\bigcap b : b$ a branch in $T\}$ is an uncountable closed set in $R^2$;*

(iii) *$\theta: T \to [Fn(\omega_1, 2)]^{\leq\omega}$ is such that $\sigma \in \theta(t) \Rightarrow \tau \subset \sigma$.*

(iv) *Let $A(t) = \bigcup_{\sigma \in \theta(t)} \operatorname{dom}(\sigma \backslash \tau)$, $A(b) = \bigcup_{t \in b} A(t)$, and $\theta(b) = \bigcup_{t \in b} \theta(t)$. If $b$ is any branch of $T$, then there exists $f \in 2^{A(b)}$ such that $f$ is incompatible with every member of $\theta(b)$.*

*Then there are $\omega_1$ many branches $b$ in $T$ such that, for some $f_b \in 2^{A(b)}$ incompatible with every member of $\theta(b)$, we have*

$$\bigcap b \in U(\tau) \cap \bigcap_{\zeta \in A(b)} U_{\zeta f_b(\zeta)}.$$

*Proof.* Let $\{C_\alpha\}_{\alpha<\omega_1}$ index all countable subsets of $R^2$ with uncountable closures, and let $\{(\tau_\alpha, T_\alpha, \theta_\alpha)\}_{\alpha<\omega_1}$ index all triples satisfying (5)(i)–(iv). We may assume that $\tau_\alpha \in Fn(\alpha, 2)$ and that $\sigma \in \theta_\alpha(t)$ implies $\sigma \in Fn(\alpha, 2)$. Inductively define countable sets $U_{\beta\gamma 0}$, $U_{\beta\gamma 1}$, where $\beta$, $\gamma < \omega_1$, satisfying:

(a) $U_{\beta\gamma 0} \cup U_{\beta\gamma 1} \subset R^2 \backslash A_\beta$, and $U_{\beta\gamma 0} \cap U_{\beta\gamma 1} = \varnothing$;

(b) $|\{p_\beta, q_\beta\} \cap U_{\beta 0 e}| = 1$ for $e < 2$;

(c) $\delta < \gamma \Rightarrow U_{\beta\delta e} \subset U_{\beta\gamma e}$;

(d) $U_{\beta 00} \cup U_{\beta 01} \supset [\bigcup_{\zeta<\beta}(A_\zeta \cup C_\zeta)] \backslash A_\beta$;

(e) $\beta < \gamma$ and $\sigma \in Fn(\gamma+1 \backslash \beta, 2)$ implies $\mathrm{cl}_{\mathscr{E}}(C_\beta \cap \bigcap_{\zeta \in \operatorname{dom} \sigma} U_{\zeta 0 \sigma(\zeta)}) \supset \{x : x$ is an uncountable limit point of $\mathrm{cl}_{\mathscr{E}}(C_\beta)\}$;

(f) if $\beta < \gamma$ and $A \subset A_\beta$ is a finite Boolean combination of $\{A_\beta\} \cup \{U_{\zeta\gamma e} : \beta \leq \zeta < \gamma,\ e < 2\}$ and $|A \backslash A_\gamma| = \omega$, then for $e < 2$, $|U_{\gamma 0 e} \cap A| = \omega$; and

(g) if $\beta < \gamma$, there exists a branch $b$ in $T_\beta$ and $f_b \in 2^{A(b)}$ incompatible with every $\sigma \in \theta(b)$ such that $\bigcap b \in \bigcap_{\delta \in \operatorname{dom} \tau_\beta} U_{\delta\gamma\tau_\beta(\delta)} \cap \bigcap_{\zeta \in A(b)} U_{\zeta\gamma f_b(\zeta)}$.

Let $\{U_{000}, U_{001}\}$ be any pair of disjoint countable subsets of $R^2 \backslash A_0$ satisfying (b). Then (a)–(g) are satisfied for $\beta = \gamma = 0$. Suppose that $U_{\beta\gamma e}$'s have been defined satisfying (a)–(g) for all $\beta$, $\gamma < \alpha$. We need to define them for $\beta$, $\gamma \leq \alpha$. First we define $U_{\beta\alpha e}$ for $\beta < \alpha$. If $\alpha$ is a limit ordinal, let $U_{\beta\alpha e} = \bigcup_{\gamma<\alpha} U_{\beta\gamma e}$. If $\alpha = \delta + 1$, $\delta \geq \omega$, pick a branch $b$ in $T_\delta$ such that

$$\bigcap b \notin \bigcup_{\beta, \gamma<\alpha} U_{\beta\gamma e} \cup \bigcup_{\beta<\alpha} A_\beta .$$

Let $A(t)$, $A(b)$, and $\theta(b)$ be as defined in (5)(iv) with $(\tau, T, \theta) = (\tau_\delta, T_\delta, \theta_\delta)$. Note that $A(b) \subset \alpha$ by our assumption that $\sigma \in \theta_\alpha(t)$ implies $\sigma \in Fn(\alpha, 2)$. Pick $f_b \in 2^{A(b)}$ incomparable with every member of $\theta(b)$. For each $\beta \in A(b)$, let

$$U_{\beta\alpha f_b(\beta)} = U_{\beta\delta f_b(\beta)} \cup \left\{\bigcap b\right\},$$

and for each $\zeta \in \operatorname{dom} \tau_\delta$, let

$$U_{\zeta\alpha\tau_\delta(\zeta)} = U_{\zeta\delta\tau_\delta(\zeta)} \cup \left\{\bigcap b\right\}.$$

(Note that $\operatorname{dom} \tau_\delta \cap A(b) = \varnothing$, so we have not added $\bigcap b$ to both $U_{\beta\delta 0}$ and $U_{\beta\delta 1}$.)

Otherwise, let $U_{\beta\alpha e} = U_{\beta\delta e}$.

Now we define $U_{\alpha 0 e}$, $e < 2$. Let $\{X_n\}_{n<\omega}$ enumerate all sets $A$ satisfying $(f)$ with $\gamma = \alpha$ and all sets of the form $C_\beta \cap \bigcap_{\zeta \in \operatorname{dom}\sigma} U_{\zeta 0 \sigma(\zeta)}$ with $\beta < \alpha$ and $\sigma \in Fn(\alpha \backslash \beta, 2)$, with each set listed infinitely often. Let $\{B_{\beta k}\}_{k<\omega}$ be a countable base in $R^2$ for condensation points of $\operatorname{cl}_{\mathscr{E}}(C_\beta)$. Let $x_0 = p_\alpha$, and $y_0 = q_\alpha$, and choose

$$x_n, y_n \in X_n \backslash (A_\alpha \cup \{x_i, y_i\}_{i<n})$$

such that $x_n \neq y_n$ and, if $X_n$ is the $k$th appearance of $C_\beta \cap \bigcap_{\zeta \in \operatorname{dom}\sigma} U_{\zeta 0 \sigma(\zeta)}$, then $x_n$, $y_n \in B_{\beta n}$. (Note that this is possible because $A_\alpha$ has at most one limit point.) Let $U_{\alpha 00} = \{x_n\}_{n<\omega}$, and let

$$U_{\alpha 01} = \left[\bigcup_{\zeta<a} (A_\zeta \cup C_\zeta)\right] \backslash (A_\alpha \cup U_{\alpha 00}).$$

Let $U_{\alpha\beta e} = U_{\alpha 0 e}$ for $\beta \leq \alpha$. It is easy to check that (a)–(g) now hold for $\beta$, $\gamma \leq \alpha$. (Note that by the inductive assumption, the essential case for (g) is $\gamma = \alpha$ and $\beta = \delta$, where $\alpha = \delta + 1$, which was taken care of above.)

Let $U_{\alpha 0} = \bigcup_{\gamma<\omega_1} U_{\alpha\gamma 0}$, $U_{\alpha 1} = R^2 \backslash (A_\alpha \cup U_{\alpha 0})$. Obviously $U_{\alpha 1} \supset \bigcup_{\gamma<\omega_1} U_{\alpha\gamma 1}$.

Clearly (1) holds. Condition (3) is clear from (f), once one notices that by (d) intersections and complements of $U_{\zeta e}$'s, $\beta \leq \zeta < \gamma$, $e < 2$, with $A_\beta$ are determined by the $U_{\zeta 0 e}$'s. Condition (4) is clear from (e).

To see that (5) holds, let $(\tau, T, \theta)$ satisfy the hypotheses of (5). It is clear from (g) that there is at least one branch $b$ in $T$ satisfying the conclusion of (5). Suppose there were only countably many. Then there is an uncountable closed subset $H$ of $\{\bigcap b : b \text{ branch in } T\}$ such that $\bigcap b \notin H$ for any branch satisfying the conclusion of (5). Let $T'$ be the subtree of $T$ consisting of all $t \in T$ which meet $H$. Then $(\tau, T', \theta \restriction T')$ satisfies the hypotheses of (5), so there exists a branch $b'$ in $T'$ satisfying the conclusion of (5), a contradiction.

We will show that (2) holds by showing that (5) implies (2). Let $f \in 2^\alpha$, $\alpha < \omega_1$, and let $K$ be an uncountable closed subset of $R^2$. Let $T$ be a tree satisfying (5)(ii) with $\bigcap b \in K$ for each branch $b$. For each $t \in T$, let $\theta(t) = \{\sigma \in Fn(\alpha, 2): \sigma$ is incompatible with $f\}$. Let $b$ and $f_b$ satisfy the conclusion of (5) for $(\varnothing, T, \theta)$. Note that $A(b) = \alpha$ and $f_b = f$. Thus

$$\bigcap b \in \bigcap_{\zeta<\alpha} U_{\zeta f(\zeta)} \cap K. \quad \square$$

**Example 3.2** (CH). There is a perfectly normal hereditarily separable (and non-Lindelöf) space in which the nondegenerate connected sets are precisely the cofinite sets.

*Proof*. Our desired space is the set $R^2$ with the topology generated by certain $U_{\alpha e}$'s obtained from Lemma 3.1, together with a certain refinement of the Euclidean topology which we now define.

Let $R^2 = \{x_\alpha\}_{\alpha<\omega_1}$, and let $\{D_\alpha\}_{\alpha<\omega_1}$ index all countable subsets of $R^2$ with uncountable closure. For each $\alpha < \omega_1$, let $S_\alpha = \bigcup_{n<\omega} S_{\alpha,n}$ be a sequence converging to $x_\alpha$ such that the $S_{\alpha,n}$'s are disjoint subsequences of $S_\alpha$ and

$$\beta < \alpha \text{ and } x_\alpha \in \overline{D}_\beta \Rightarrow \forall n(|S_{\alpha,n} \cap D_\beta| = \omega).$$

This can easily be done, since for each $\alpha$ there are only countably many $D_\beta$'s to worry about.

Now call $B$ an open neighborhood of $x_\alpha$ if $x_\alpha \in B$, $B\backslash\{x_\alpha\}$ is Euclidean open, and $B$ contains a tail of all but finitely many of the $S_{\alpha,n}$'s. Let $\mathscr{T}$ be the topology on $R^2$ generated by these $B$'s. It is easy to check that $\mathscr{T}$ is regular and that every infinite subset of $R^2$ contains a $\mathscr{T}$-closed discrete infinite subset.

We show that $\mathscr{T}$ is (hereditarily) Lindelöf and, hence, completely regular. Suppose not; then there is subset $Y = \{y_\alpha\}_{\alpha<\omega_1}$ of $X$ which is right-separated, i.e., $y_\alpha \notin \mathrm{cl}_{\mathscr{T}}\{y_\beta : \beta > \alpha\}$ for each $\alpha < \omega_1$. We may assume $Y$ is uncountably dense-in-itself in the Euclidean topology. Let $D \subset Y$ be a countable Euclidean-dense subset. Then there exists $\beta < \omega_1$ such that $D = D_\beta$, and there exists $\alpha > \beta$ with $x_\alpha \in Y$. Then every $\mathscr{T}$-neighborhood of $x_\alpha$ contains an $\mathscr{E}$-neighborhood of a point in $D$ and, hence, contains uncountably many $y \in Y$, a contradiction.

Let $\{(A_\alpha, \{p_\alpha, q_\alpha\})\}_{\alpha<\omega_1}$ list all pairs $(A, \{p, q\})$ such that $A$ is a countably infinite $\mathscr{T}$-closed-discrete subset of $R^2$, $A^\cdot$ has at most one Euclidean limit point, and $p, q \in R^2\backslash A$. Let $\{U_{\alpha 0}, U_{\alpha 1}\}_{\alpha<\omega_1}$ be as in Lemma 3.1, and let $\Sigma$ be the topology generated by the $U(\sigma)$'s, $\sigma \in Fn(\omega_1, 2)$. Let $\rho$ be the topology generated by $\Sigma \cup \mathscr{T}$. We claim that $(R^2, \rho)$ satisfies the conditions of Example 3.2.

First we show that $(R^2, \rho)$ is completely regular. Let $x \in U(\sigma) \cap B$, $B$ $\mathscr{T}$-open. Without loss of generality, $\overline{B}^{\mathscr{T}} \cap (\bigcup_{\alpha\in\mathrm{dom}\,\sigma} A_\alpha) = \varnothing$ (since the $A_\alpha$'s are $\mathscr{T}$-closed and $x$ being in $U(\sigma)$ implies $x \notin \bigcup_{\alpha\in\mathrm{dom}\,\sigma} A_\alpha$). Since $\mathscr{T}$ is completely regular, there is a $\mathscr{T}$-continuous $f: R^2 \to [0, 1]$ such that $f(R^2\backslash B) = 1$ and $f(x) = 0$. Define $\overline{f}$ by

$$\overline{f}(y) = \begin{cases} f(y) & \text{if } y \in U(\sigma) \cap B, \\ 1 & \text{otherwise}. \end{cases}$$

It is not difficult to check that $\overline{f}$ is $\rho$-continuous. It follows that $(R^2, \rho)$ is completely regular.

Now we show that $\rho$ is hereditarily separable. Suppose on the contrary that $Y = \{y_\alpha\}_{\alpha<\omega_1}$ is left-separated. Let $y_\alpha \in U(\sigma_\alpha) \cap B_\beta$, where $B_\alpha \in \mathscr{T}$, such that $\beta < \alpha$ implies $y_\beta \notin U(\sigma_\alpha) \cap B_\alpha$. Without loss of generality, $Y$ is uncountably dense-in-itself in $\mathscr{E}$, and the $\sigma_\alpha$'s form a $\Delta$-system with root $\Delta$. Note that $Y \subset U(\Delta)$. Let $C$ be a countable $\mathscr{E}$-dense subset of $Y$. By Lemma 3.1(4), there is $\alpha_0 < \omega_1$ such that $U(\sigma) \cap C$ is dense in $C$ for each $\sigma \in Fn(\omega_1 \backslash \alpha_0, 2)$. There exists $\alpha_1 \geq \alpha_0$ such that $\operatorname{dom}(\sigma_\alpha \backslash \Delta) \subset \omega_1 \backslash \alpha_0$ for each $\alpha \geq \alpha_1$. Then $U(\sigma_\alpha) \cap C$ is dense in $C$ for each $\alpha \geq \alpha_1$ since $C \subset U(\Delta)$. Let $\delta$ be such that $C = D_\delta$, and pick $\alpha > \alpha_1 + \sup\{\beta : y_\beta \in C\}$ such that $y_\alpha = x_\mu$ for some $\mu > \delta$. Then $y_\alpha$ is a $\mathscr{T}$-limit to $C$; indeed every $\mathscr{T}$-neighborhood of $y_\alpha$ contains an $\mathscr{E}$-neighborhood of a point in $C$. Thus $C \cap U(\sigma_\alpha) \cap B_\alpha \neq \varnothing$, contradicting $y_\beta \notin U(\sigma_\alpha) \cap B_\alpha$ for $\beta > \alpha$.

Let $O$ be relatively clopen in $R^2 \backslash F$, $F$ finite. Suppose $\tau \in Fn(\omega_1, 2)$, $E$ is Euclidean open, and $U(\tau) \cap E \cap (R_2 \backslash F) \subset O$. We aim to show $U(\tau) \cap (R^2 \backslash F) \subset O$.

We may assume that $E$ is maximal in the sense that $U(\tau) \cap E' \cap (R^2 \backslash F) \not\subset O$ for any $\mathscr{E}$-open $E' \not\supseteq E$. Suppose $E$ is dense in an $\mathscr{E}$-open $G$. Then $x \in U(\tau) \cap G \Rightarrow x \in \overline{(U(\tau) \cap E)}^\rho$ (since $x \in U(\sigma) \cap B$, $B \in \mathscr{T}$, implies $\sigma$ is compatible with $\tau$ and $B \cap E$ contains an $\mathscr{E}$-open subset, so $U(\sigma) \cap B \cap U(\tau) \cap E \supset U(\sigma \cup \tau) \cap (B \cap E) \neq \varnothing$). Thus $U(\tau) \cap G \cap (R^2 \backslash F) \subset O$, so $G \subset E$ by maximality of $E$. Therefore, $E$ is regular open.

Suppose that $U(\tau) \cap (R^2 \backslash F) \not\subset O$. Then $E$ is a proper regular open subset of $R^2$, whence, $\partial E$ contains an uncountable Cantor set $K$ missing $F$. Let $T$ be a Cantor tree of basic $\mathscr{E}$-open sets such that each branch of $T$ is a neighborhood base for some unique $x \in K$, and $K = \{\bigcap b : b \text{ a branch in } T\}$. If $\sigma \supset \tau$ and $t \in T$, call $\sigma$ *$t$-minimal* if $U(\sigma) \cap (t \backslash F) \subset O$, but $U(\sigma') \cap (t \backslash F) \not\subset O$ for any $\sigma'$ with $\tau \subset \sigma' \subsetneqq \sigma$. Note that by our assumption that $E$ is maximal, $U(\tau) \cap (t \backslash F) \not\subset O$ for any $t \in T$.

For $t \in T$, let $\theta(t) = \{\sigma \supset \tau : \sigma \text{ is } t\text{-minimal}\}$. We need to show $|\theta(t)| \leq \omega$. If not, there is an infinite $\Delta$- system $R \subset \theta(t)$ with root $r$. Let $x \in U(r) \cap (t \backslash F)$, and let $U(\sigma_x) \cap B_x$ be a basic $\rho$-neighborhood of $x$. Since $r \supset \tau$, $\sigma_x$ is compatible with $\tau$ and, hence, with some $\sigma \in R$. Then $(U(\sigma_x) \cap B_x) \cap (U(\sigma) \cap (t \backslash F)) \supset U(\sigma_x \cup \sigma) \cap (B_x \cap (t \backslash F)) \neq \varnothing$. Thus $x$ is in the closure of $\bigcup\{U(\sigma) \cap (t \backslash F) : \sigma \in R\} \subset O$, so $x \in O$. Thus $U(r) \cap (t \backslash F) \subset O$, contradicting the $t$-minimality of the $\sigma$'s in $R$.

Suppose that $(\tau, T, \theta)$ as defined above satisfies (5)(i)–5(iv) of Lemma 3.1. For each $x \in K$, let $b(x) = \{t \in T : x \in t\}$. By the conclusion of condition (5), the subset $X$ of $K$ consisting of all $x \in K$ with

$$x = \bigcap b(x) = \bigcap_{\zeta \in A(b)} U_{\zeta f_x(\zeta)} \cap U(\tau)$$

for some $f_x \in 2^{A(b(x))}$ incompatible with every $\sigma \in \theta(b(x))$ is uncountable. Let $C$ be a countable $\rho$-dense subset of $X$, and let $x \in X \backslash C$. Since $x \in \overline{C}^{\mathscr{T}}$ and $C \subset \delta E$, it follows that $x \in \overline{E}^{\mathscr{T}}$. Since also $x \in U(\tau)$, we must have $x \in \overline{U(\tau) \cap E}^\rho$, hence $x \in O$. Let $x \in U(\sigma) \cap (B \backslash F) \subset O$, $B$ a $\mathscr{T}$-basic open set, and $\sigma \supset \tau$. Choose $y \in C \cap U(\sigma) \cap (B \backslash F)$ and $t \in b(y)$ with $t \subset B$.

Then $y \in U(\sigma) \cap (t \backslash F) \subset O$, so $\sigma' \in \theta(t)$ for some $\sigma'$ with $\tau \subset \sigma' \subset \sigma$. Then $y \in \bigcap_{\zeta \in \operatorname{dom} \sigma'} U_{\zeta \sigma'(\zeta)} \cap \bigcap_{\zeta \in A(b(y))} U_{\zeta f_y(\zeta)}$, contradicting $f_y$ incompatible with $\sigma'$.

We have thus shown that $(\tau, T, \theta)$ does not satisfy conditions (5)(i)–(iv) of Lemma 3.1. Thus it is (5)(iv) that is not satisfied, i.e., there is a branch $b$ in $T$ such that no $f \in 2^{A(b)}$ is incompatible with every $\sigma \in \theta(b)$. For each $\sigma \in \theta(b)$, let $[\sigma] = \{f \in 2^{A(b)} : \sigma \backslash \tau \subset f\}$. Then the $[\sigma]$'s cover $2^{A(b)}$, so there is a finite $S \subset \theta(b)$ with $\bigcup\{[\sigma] : \sigma \in S\} = 2^{A(b)}$. Let $t \in b$ such that $S \subset \bigcup\{\theta(t') : t' \in b, t' \supset t\}$. Suppose $x \in U(\tau) \cap (t \backslash F)$, $x \notin \bigcup_{\alpha \in A(b)} A_\alpha$. Define $f_x \in 2^{A(b)}$ by $f_x(\alpha) = e$ iff $x \in U_{\alpha e}$. Let $\sigma \in S$ such that $f_x \in [\sigma]$. Then $x \in U(\sigma) \cap (t \backslash F) \subset O$. So $U(\tau) \cap (t \backslash F) \backslash O$ is at most countable, so $U(\tau) \cap (t \backslash F) \cap O$ is $\rho$-dense in $U(\tau) \cap (t \backslash F)$ and, hence, $U(\tau) \cap (t \backslash F) \subset O$, contradicting the maximality of $E$. This completes the proof that $U(\tau) \cap (E \backslash F) \subset O$ for $E \in \mathscr{E}$ and $F$ finite implies $U(\tau) \backslash F \subset O$.

It follows that any proper $\rho$-clopen partition of $R^2 \backslash F$, $F$ finite, is a $\Sigma$-clopen partition. By Lemma 1.1, there are no such partitions. Hence cofinite subsets of $R^2$ are $\rho$-connected.

Let $Z$ be any coinfinite subset of $R^2$ and $p \neq q \in R^2 \backslash Z$. There is a countably infinite $\mathscr{T}$-closed discrete $A \subset R^2 \backslash Z$. Then $(A, \{p, q\}) = (A_\alpha, \{p_\alpha, q_\alpha\})$ for some $\alpha$, so $\{Z \cap U_{\alpha 0}, Z \cap U_{\alpha 1}\}$ is a relatively $\rho$-clopen partition of $Z$ separating $p$ and $q$. Thus every coinfinite subset of $R^2$ is totally disconnected.

We now show that $(R^2, \rho)$ is perfectly normal. Let $H$ be $\rho$-closed; we need to prove that $H$ is a regular $G_\delta$ set.

Let $\mathscr{B}$ be a countable base for $R^2$, and for $B \in \mathscr{B}$, let $\Sigma(B)$ be all $\sigma \in Fn(\omega_1, 2)$ such that

(i) $\operatorname{cl}_{\mathscr{E}}(B) \cap (\bigcup\{A_\alpha : \alpha \in \operatorname{dom} \sigma\}) = \varnothing$;
(ii) $\overline{U(\sigma) \cap B}^\rho \cap H$ has countable $\mathscr{E}$-closure;
(iii) $\tau \subsetneq \sigma$ implies $\tau$ does not satisfy (ii).

We first claim that $|\Sigma(B)| \leq \omega$. If not, there is an uncountable $\Delta$-system $T \subset \Sigma(B)$ with root $\Delta$. Then by (iii), $C = \overline{U(\Delta) \cap B}^\rho \cap H$ has uncountable $\mathscr{E}$-closure. Let $\alpha$ be as in Lemma 3.1(4) for $C$. Let $\sigma \in T$ be such that $\operatorname{dom}(\sigma \backslash \Delta) \subset \omega_1 \backslash \alpha$. Then

$$\operatorname{cl}_{\mathscr{E}}(U(\sigma \backslash \Delta) \cap C) \supset \{x : x \text{ is an uncountable limit point of } \operatorname{cl}_{\mathscr{E}}(C)\};$$

whence, $[U(\sigma \backslash \Delta) \cap C] \cap H$ has uncountable $\mathscr{E}$-closure. Now $C \subset \overline{U(\Delta) \cap B}^\rho \subset U(\Delta) \cap \operatorname{cl}_{\mathscr{E}}(B) \subset U(\Delta)$ by (i), so $U(\sigma \backslash \Delta) \cap C = U(\sigma) \cap C \subset \overline{U(\sigma) \cap B}^\rho$. Thus $\overline{U(\sigma) \cap B}^\rho \cap H$ has uncountable $\mathscr{E}$-closure, contradicting (ii).

Let $\mathscr{J} = \{U(\sigma) \cap B : \sigma \in \Sigma(B), B \in \mathscr{B}\}$. We claim that $\mathscr{J}$ covers all but countably many points of $R^2 \backslash H$. Suppose $X \subset R^2 \backslash (H \cup \bigcup \mathscr{J})$ is uncountable, and let $x \in X$ be a $\rho$-limit point of $X$. There is a basic $\mathscr{T}$- neighborhood $N_x$ of $x$ and $\sigma \in Fn(\omega_1, 2)$ such that $x \in N_x \cap U(\sigma) \subset \overline{N_x \cap U(\sigma)}^\rho \subset R^2 \backslash H$. Without loss of generality, $N_x \cap (\bigcup\{A_\alpha : \alpha \in \operatorname{dom} \sigma\}) = \varnothing$. Let $y \in X \cap N_x \cap U(\sigma)$, $y \neq x$, and let $y \in B \in \mathscr{B}$ with $\operatorname{cl}_{\mathscr{E}}(B) \subset N_x$. Then $\sigma$ and $B$ satisfy (i) and (ii) above, so $\sigma' \in \Sigma(B)$ for some $\sigma' \subset \sigma$. Hence $y \in U(\sigma') \cap B \in \mathscr{J}$, a contradiction.

For $\sigma \in \Sigma(B)$, let $K(\sigma, B) = \overline{U(\sigma) \cap B}^\rho \cap H$. Each $K(\sigma, B)$ has countable $\mathscr{E}$-closure and is $\rho$-closed and, hence, is a regular $G_\delta$-set in $\rho$. Say $K(\sigma, B) =$

$\bigcap_{n<\omega} O_n(\sigma, B) = \bigcap_{n<\omega} \overline{O_n(\sigma, B)}^{\rho}$, where $O_n(\sigma, B)$ is $\rho$-open. Then

$$\{(U(\sigma) \cap B) \backslash \overline{O_n(\sigma, B)}^{\rho} : n < \omega,\ \sigma \in \Sigma(B),\ B \in \mathscr{B}\}$$

is a countable collection of $\rho$-open sets whose $\rho$-closures miss $H$, and this collection covers all but countably many points of $R^2 \backslash H$. It follows that $H$ is a regular $G_\delta$-set in $\rho$.

Finally, we show that $(R^2, \rho)$ is not Lindelöf. For each $\alpha < \omega_1$, choose $\delta(\alpha) > \{\delta(\beta) : \beta < \alpha\}$ with $x_\alpha \in U_{\delta(\alpha)e(\alpha)}$. It follows from Lemma 3.1(2) that $\mathscr{U} = \{U_{\delta(\alpha)e(\alpha)}\}_{\alpha<\omega_1}$ is an open cover with no countable subcover. □

## 4. Variations

In this section we indicate how to modify the previous constructions to obtain examples under Martin's Axiom. We also discuss examples in which the connected sets are co-$< \lambda$ sets for cardinals $\lambda > \omega$. Finally, we mention a number of open questions.

The CH construction of a countable Hausdorff example can easily be modified to be obtained under a certain "small cardinal" hypothesis. A family $\mathscr{F}$ of infinite subsets of $\omega$ is said to be reaped by a set $R \subset \omega$ if $|F \cap R| = |F \backslash R| = \omega$ for every $F \in \mathscr{F}$. The *reaping number* or *refinement number* $\mathfrak{r}$ is defined to be the least cardinal of a family $\mathscr{F} \subset [\omega]^\omega$ that cannot be reaped by any $R \subset \omega$. Of course, $\omega < \mathfrak{r} \leq \mathfrak{c}$, and it is known that MA implies $\mathfrak{r} = \mathfrak{c}$. (This fact also follows from Lemma 4.2) See [V] for a summary of results about $\mathfrak{r}$.

**Example 4.1** ($\mathfrak{r} = \mathfrak{c}$). There is a countable Hausdorff space in which the nondegenerate connected sets are precisely the co-finite sets.

*Proof*. The proof is the same as the proof is Example 2.1, with an induction of length $\mathfrak{c}$ instead of $\omega_1$. Note that to construct the desired partition of $Q \backslash A_\alpha$, a collection of fewer than $\mathfrak{c}$-many sets needed to be reaped. If $\mathfrak{r} = \mathfrak{c}$, this can always be done. □

Essentially two things are necessary to modify Example 3.2 to obtain a completely regular example under MA. One is that collections of fewer than $\mathfrak{c}$-many countable subsets of a set $X$ need to be reaped, where $X$ may be uncountable. The other is to construct an appropriate topology on $R^2$. The next three lemmas take care of these tasks.

**Lemma 4.2** (MA). *Let $\mathscr{A}$ be a collection of countably infinite subsets of a set $X$, with $|\mathscr{A}| < \mathfrak{c}$. Then there exists a set $Y \subset X$ such that $|A \cap Y| = |A \backslash Y| = \omega$ for every $A \in \mathscr{A}$.*

*Proof*. Let $P$ be the set of all finite sequences $p = \langle (Y_A^0(p), Y_A^1(p)) \rangle_{A \in \mathscr{F}(p)}$, where $\mathscr{F}(p) \in [\mathscr{A}]^{<\omega}$ and $Y_A^i(p) \in [A]^{<\omega}$ for all $A \in \mathscr{F}(p)$ and $i < 2$, such that

$$\left( \bigcup_{A \in \mathscr{F}(p)} Y_A^0(p) \right) \cap \left( \bigcup_{A \in \mathscr{F}(p)} Y_A^1(p) \right) = \varnothing.$$

Define $q \leq p$ iff $\mathscr{F}(q) \supset \mathscr{F}(p)$ and $Y_A^i(q) \supset Y_A^i(p)$ for each $A \in \mathscr{F}(p)$. A standard $\Delta$-system argument shows that $P$ is ccc.

For $A \in \mathscr{A}$ and $n < \omega$, let

$$D(A, n) = \{p \in P : A \in \mathscr{F}(p) \text{ and } |Y_A^i(p)| \geq n \text{ for } i < 2\}.$$

Let $G \subset P$ meet each $D(A, n)$. Then $Y = \bigcup\{Y_A^0(p) : p \in G, A \in \mathscr{F}(p)\}$ satisfies the conclusion of the lemma. □

**Lemma 4.3** (MA). *Let $M$ be a regular space with a countable base $\mathscr{B}$. Let $\{D_\alpha\}_{\alpha<\kappa}$ be a collection of a subsets of $M$, $\kappa < \mathfrak{c}$, and let $x \in M$ be such that $x \in \overline{D_\alpha \backslash \{x\}}$ for each $\alpha < \kappa$.*

*Then there are $B_{nm} \in \mathscr{B}$, $n$, $m < \omega$, satisfying:*

(a) $\overline{B}_{nm} \cap \overline{B}n'm' = \varnothing$ *if* $(n, m) \neq (n', m')$;

(b) *every neighborhood of $x$ contains all but finitely many of the $B_{nm}$'s;*

(c) $x \notin \overline{B}_{nm}$ *for every* $n$, $m < \omega$;

(d) *for each* $\alpha < \kappa$ *and* $n < \omega$, $|\{m : B_{nm} \cap D_\alpha \neq \varnothing\}| = \omega$.

*Proof*. Let $\{E_n(x)\}_{n<\omega}$ be a countable decreasing base at $x$. Let $P$ be the set of all finite sequences $p = \langle B_i(p)\rangle_{i<n(p)}$ satisfying:

(i) $B_i(p) \in \mathscr{B}$ for each $i < n(p)$;

(ii) $i \neq j < n(p)$ implies $\overline{B_i(p)} \cap \overline{B_j(p)} = \varnothing$;

(iii) $B_i(p) \subset E_i(p)$ and $x \notin \overline{B_i(p)}$ for each $i < n(p)$.

Order $P$ by extension.

$P$ is countable and, hence, ccc. Let $\{Z_n\}_{n<\omega}$ be a partition of $\omega$ into infinite sets. For $\alpha < \kappa$ and $n$, $k < \omega$, let

$$D_{\alpha nk} = \{p \in P : \exists i < n(p)\,(i > k \wedge i \in Z_n \wedge B_i(p) \cap D_\alpha \neq \varnothing)\}\,.$$

Each $D_{\alpha nk}$ is dense in $P$. Let $G$ be a filter meeting all the $D_{\alpha nk}$'s. Then $G$ defines a sequence $\langle B_i\rangle_{i<\omega}$ of elements of $\mathscr{B}$. Let $\{B_{nm}\}_{m<\omega}$ enumerate $\{B_i\}_{i\in Z_n}$. It is easy to check that this works. □

A *butterfly topology* on $R^2$ is a topology such that each point $x \in R^2$ has a basis consisting of sets $N$ with $N\backslash\{x\}$ Euclidean open.

**Lemma 4.4** (MA). *There is a completely regular butterfly topology on $R^2$ with the density $d(X) < \mathfrak{c}$ for every $X \subset R^2$ and such that every infinite set contains an infinite closed discrete set.*

*Proof*. Let $R^2 = \{x_\alpha\}_{\alpha<\mathfrak{c}}$, and let $\{D_\alpha\}_{\alpha<\mathfrak{c}}$ enumerate all countable dense-in-themselves subsets of $R^2$. Let $\mathscr{B}$ be the standard countable base for $R^2$, and let $\{B_{nm}(\alpha): n, m < \omega\}$ be the subset of $\mathscr{B}$ guaranteed to exist by Lemma 4.3 applied to $x_\alpha$ and $\{D_\beta : \beta < \alpha \text{ and } x_\alpha \in \overline{D}_\beta\}$. Declare $U \cup \{x_\alpha\}$ to be an open neighborhood of $x_\alpha$ if $U$ is Euclidean open and for sufficiently large $n < \omega$, $U \supset B_{nm}$ for sufficiently large $m < \omega$. It is easy to check that the resulting topology $\mathscr{T}$ on $R^2$ has hereditary density less than $\mathfrak{c}$ and that every infinite subset of $R^2$ contains an infinite $\mathscr{T}$-closed discrete subset.

To see that $(R^2, \mathscr{T})$ is completely regular, let $N$ be any $\mathscr{T}$- neighborhood of a point $x_\alpha$, where $N\backslash\{x_\alpha\}$ is Euclidean open. Since $\mathscr{B}' = \{\overline{B}_{nm}(\alpha) : \overline{B}_{nm}(\alpha) \subset N,\ n, m < \omega\}$ is Euclidean closed discrete in $R\backslash\{x_\alpha\}$, there is a Euclidean open $U \subset N$ such that $U \supset \bigcup\mathscr{B}'$, and if $U_{nm}$ is the component of $U$ containing $\overline{B}_{nm}$, then $\overline{U}_{nm} \subset U$ and $\{U_{nm} : \overline{B}_{nm}(\alpha) \in \mathscr{B}'\}$ is a discrete collection of sets in $R^2\backslash\{x_\alpha\}$. There exist Euclidean continuous functions $f_{nm}: R^2 \to [0, 1]$ with $f_{nm}(\overline{B_{nm}(\alpha)}) = 1$ and $f_{nm}(R^2\backslash U_{nm}) = 0$. Define $f: R^2 \to [0, 1]$ by $f(x_\alpha) = 1$ and $f(x) = \sum_{n,m<\omega} f_{nm}(x)$ if $x \neq x_\alpha$. It is easy to check that $f$ is $\mathscr{T}$-continuous. Since $f(x_\alpha) = 1$ and $f(R^2\backslash N) = 0$, this proves that $(R^2, \mathscr{T})$ is completely regular. □

**Example 4.5** (MA). There is a completely regular space in which the nondegenerate connected sets are precisely the cofinite sets.

*Proof* (*Outline*). First note that the analogue of Lemma 3.1, with $\omega_1$ replaced by $\mathfrak{c}$, holds under MA. This is proven in the same way, by an induction of length $\mathfrak{c}$, with the $U_{\beta\gamma e}$'s having size less than $\mathfrak{c}$ at each stage. To define the $U_{\alpha 0e}$'s, a collection of fewer than $\mathfrak{c}$-many countable sets needs to be reaped; this is where Lemma 4.2 is used.

Now follow the proof of Example 3.2, using the topology on $R^2$ given by Lemma 4.4 in place of $\mathscr{T}$. The topology $\rho$ will not be hereditarily separable (if $\mathfrak{c} > \omega_1$) but will be hereditarily $(< \mathfrak{c})$-separable, and that suffices. □

Now we take up the problem of obtaining spaces in which the connected sets are co-$< \lambda$, i.e., their complements have cardinality less than $\lambda$, for cardinals $\lambda > \omega$. It is not difficult to use Lemma 1.1 to construct Hausdorff examples for any $\lambda$ under the assumption $2^\lambda = \lambda^+$.

**Example 4.6** $(2^\lambda = \lambda^+)$. A Hausdorff space of cardinality $\lambda^+$ in which the nondegenerate connected sets are precisely the (co-$< \lambda$)-sets.

*Proof* (*Outline*). Let $|X| = \lambda^+$, and let $\{(A_\alpha, \{p_\alpha, q_\alpha\})\}_{\alpha<\lambda^+}$ index all pairs $(A, \{p, q\})$ such that $A \in [X]^\lambda$, $p, q \notin A$. Inductively construct partitions $\{U_{\alpha 0}, U_{\alpha 1}\}$ of $X \backslash A_\alpha$ such that $|U(\sigma)| = \lambda^+$ for each $\sigma \in Fn(\lambda^+, 2)$. At stage $\alpha < \lambda^+$, one needs to

(i) make $|U_{\alpha e} \cap U(\sigma)| = \lambda^+$ for each $\sigma \in Fn(\alpha, 2)$; and

(ii) make Lemma 1.1(b) hold for $\leq \lambda$-many sets of size $\lambda$.

It is easy to check that this can be done. □

It seems to be harder to obtain Hausdorff spaces of size $\lambda$ in which the connected sets are the (co-$< \lambda$)-sets. To carry out an inductive construction of $U_{\alpha e}$'s satisfying the conditions of Lemma 1.1, it is useful to have some sense in which the $A$'s are "small" and the $U(\sigma)$'s are "large".

In Example 4.6, "small" means "cardinality $\leq \lambda$" and "large" means "cardinality $\lambda^+$". But in the problem under consideration now, small and large sets will have the same cardinality, so something else must be used. In Example 2.1 or (4.1), which takes care of the case $\lambda = \omega$, "small" means "at most one limit point" and "large" means "dense in $Q$". For a regular cardinal $\lambda > \omega$, a natural idea is to take "small" to mean "nonstationary". This works. The simplest way to choose the partitions is by Cohen forcing—this avoids messy combinatorics with stationary sets.

**Examples 4.7.** Let $M$ be a model of ZFC, and let $\lambda$ be a regular uncountable cardinal in $M$ with $2^\lambda = \lambda^+$. If $\lambda^+$ Cohen reals are added to $M$, then in the extension there is a Hausdorff space of cardinality $\lambda$ in which the nondegenerate connected sets are precisely the (co-$< \lambda$)-sets.

*Proof*. See [K] for forcing terminology. Let $M \models 2^\lambda = \lambda^+$, where $\lambda$ is a regular uncountable cardinal. Let $P = Fn(\lambda^+ \times \lambda, 2)$, and let $G$ be a $P$-generic filter. Note that $M[G] \models 2^\lambda = \lambda^+$.

In $M[G]$, let $\{A_\alpha\}_{\alpha<\lambda^+}$ list all nonstationary subsets of $\lambda$ cardinality $\lambda$, such that $A_\alpha = A_{\alpha+n}$ for $n < \omega$. For each $\alpha < \lambda^+$, there is $\delta(\alpha) < \lambda^+$ such

that

$$A_\alpha \in M[G \cap Fn(\delta(\alpha) \times \lambda, 2)].$$

(See [K, VIII, Lemma 2.2].) We may assume $\alpha < \beta$ implies $\delta(\alpha) < \delta(\beta)$. Now, for each $\alpha < \lambda^+$ and $e < 2$, let

$$U_{\alpha e} = \left\{\gamma \colon \bigcup G(\delta(\alpha), \gamma) = e\right\} \backslash A_\alpha.$$

*Claim* 1. Let $\{\alpha_i\}_{i \le n}$ be a sequence of distinct elements of $\lambda$, and $e_i < 2$ for $i \le n$. Then $\bigcap_{i \le n} U_{\alpha_i e_i}$ is stationary in $\lambda$.

*Proof of Claim* 1. Suppose $n$ is least such that Claim 1 fails, and let $L = \bigcap_{i \le n} U_{\alpha_i e_i}$ be nonstationary. Then there exists a nonstationary $L' \in M$ with $L \subset L'$ (since every club in the extension contains a ground model club; see [K, VII, Exercise H1]). If $n = 0$, we have a contradiction since $U_{\alpha_0 e_0}$ clearly is not contained in a ground model nonstationary set.

So we may assume $n > 0$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then $\delta(\alpha_i) < \delta(\alpha_n)$ for each $i < n$, so $\{A_{\alpha_n}, \bigcup_{i<n} U_{a_i e_i}\} \subset M[G \cap Fn(\delta(\alpha_n) \times \lambda, 2)] \stackrel{\text{def}}{=} N$.

Let $W = \bigcap_{i<n} U_{\alpha_i e_i} \backslash (L' \cap A_{\alpha_n})$. Then $W \in [\lambda]^\lambda \cap N$. As $G \cap Fn(\{\delta(\alpha_n)\} \times \lambda, 2)$ is $Fn(\{\delta(\alpha_n)\} \times \lambda, 2)$-generic over $N$ (see [K, VIII, Theorem 2.1]), there is $\gamma \in W$ with $\bigcup G(\delta(\alpha_n), \gamma) = e_n$, whence $\gamma \in \bigcap_{i \le n} U_{\alpha_i e_i} \backslash L'$, a contradiction.

*Claim* 2. Suppose $\lambda > \beta$ and $A \subset A_\beta$ is a finite Boolean combination of $A_\beta \cup \{U_{\delta e} \colon e < 2,\ \beta \le \delta < \alpha\}$. If $|A \backslash A_\alpha| = \lambda$, then $|U_{\alpha e} \cap A| = \lambda$ for each $e < 2$.

*Proof of Claim* 2. As in the proof of Claim 1, $A \backslash A_\alpha \in M[G \cap Fn(\delta(\alpha) \times \lambda, 2)]$, whence by genericity, $\{\gamma \in A \backslash A_\alpha : \bigcup G(\delta(\alpha), \gamma) = e\}$ has cardinality $\lambda$.

It follows immediately from Claims 1 and 2 that the conditions of Lemma 1.1 are satisfied (with $X = \lambda$ and $\kappa = \lambda^+$). Hence (co-$< \lambda$)-subsets of $\lambda$ are connected in the partition topology $\Sigma$.

Since $A_\alpha = A_{\alpha+n}$ for $n < \omega$, each nonstationary $A \in [\lambda]^\lambda \cap M[G]$ is listed on an infinite ground model set of indices; hence for each $p \ne q \notin A$, there is $\alpha < \lambda^+$ such that $A_\alpha = A$ and $\bigcup G(\alpha, p) \ne \bigcup G(\alpha, q)$. It follows that $(\lambda, \Sigma)$ is Hausdorff and that $\lambda \backslash A$ is totally disconnected. Since every subset of $\lambda$ of size $\lambda$ contains a nonstationary set of size $\lambda$, the complement of any set of size $\lambda$ is totally disconnected. □

We conclude the paper by stating some open questions.

**Question 4.8.** Is there a completely regular space $X$ in which the nondegenerate connected sets are precisely the (co-$< |X|$)-sets? Or co-$< \lambda$ for some cardinal $\lambda > \omega$?

**Question 4.9.** Is there a paracompact Hausdorff (or regular Lindelöf) space in which the nondegenerate connected sets are precisely the cofinite sets?

Of course, the major open problem is:

**Question 4.10.** Is there in ZFC a Hausdorff (or completely regular) space in which the nondegenerate connected sets are precisely the cofinite sets?

## References

[E] P. Erdös, *Some remarks on connected sets*, Bull. Amer. Math. Soc. **50** (1944), 442–446.

[K] K. Kunen, *Set theory*, North-Holland, Amsterdam, 1990.

[Ts] S. F. Tsvid, *A countable strongly unicoherent space*, Mat. Zemetki **24** (1978), 655–657.

[Tz] V. Tzannes, *Three countable connected spaces*, Colloq. Math. **56** (1988), 267–279.

[V] J. E. Vaughan, *Small uncountable cardinals and topology*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990.

[W] S.Watson, *Problems I wish I could solve*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990.

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN UNIVERSITY, ALABAMA 36849-5310

*E-mail address*: GGRUEN@ducvax.auburn.edu