ON THE K-GROUPS OF CERTAIN C*-ALGEBRAS
USED IN E-THEORY

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ABSTRACT. Let $A$ be a C*-algebra. We denote by $A_\infty$ the quotient of the
C*-algebra of bounded continuous functions $[1, \infty) \to A$ by the ideal of the
functions which vanish at $\infty$. We show that the the canonical map $A \to A_\infty$
gives an isomorphism between K-groups, provided that $A$ is stable.

In [CH] the notion of asymptotic homomorphism was introduced, one of the
many remarkable achievements being a simpler description for the operations
in K-theory. Roughly speaking, modulo suspensions and stabilizations of the
E-category is obtained from the homotopy category of separable C*-algebras by
"inverting certain arrows".

In this note we shall examine one type of homomorphisms which should be
inverted in E-theory. By computing the K-groups that are involved we hope
to bring some evidence that one can expect those homomorphisms to be E-
equivalences.

To be more specific, let $A$ be a C*-algebra. We denote by $C^b([1, \infty), A)$, or
for short $C^bA$, the C*-algebra of continuous and bounded functions $f : [1, \infty)$
$\to A$. By $A_\infty$ one denotes the quotient $C^bA/CA$, where $CA$ (sometimes
denoted $C_0([1, \infty), A)$) is the ideal of all functions in $C^bA$ which vanish
at infinity. Let $i : A \to A_\infty$ be the "standard embedding" given by $i(a) =
[\text{const } a] \pmod{CA}$. Let us take, using the Bartle-Graves Theorem, a section
$p : A_\infty \to C^bA$ for the map $\pi_A : C^bA \to A_\infty$ with the following properties:

(i) $p(\lambda x) = \lambda p(x)$ for all $x \in A$, $\lambda \in \mathbb{C}$;
(ii) $p$ is continuous.

If we take $\phi^i_t : A_\infty \to A$ to be the map defined, for each $t \in [1, \infty)$, by
$\phi^i_t(x) = p(x)(t)$, then using the terminology of [CH] we get an asymptotic
homomorphism from $A_\infty$ to $A$.

It is easy to show that $\phi^i_t \circ i : A \to A$ gives an asymptotic homomorphism
which is homotopic to $\text{Id}_A$. So $i : A \to A_\infty$ has a left inverse in $E(A_\infty, A)$.
As noted in [CH], every asymptotic homomorphism $\psi_t : B \to A$ is equivalent
to one of the form $\phi^i_t \circ \psi$ with $\psi : B \to A_\infty$ a *-homomorphism.
The natural question that arises is then: For a stable C*-algebra $A$, is $\iota : A \to A_{\infty}$ an equivalence, i.e., is the asymptotic homomorphism $\iota \cdot \phi_i^0$ homotopic (possibly after further stabilizations and suspensions) to $\text{Id}_A$?

Note that $A_{\infty}$ is not separable, so this problem is somehow "out of E-theory".

In this note we shall prove the following.

**Theorem.** For stable C*-algebras $A$ the embeddings $\iota : A \to A_{\infty}$ induce isomorphisms at the level of $K$-groups.

Let us see the analogy with a more familiar situation. We know that to any extension

$$0 \to A \otimes \mathcal{H} \to B \to C \to 0$$

one can associate its *Busby invariant* which is a *-homomorphism $C \to Q(A \otimes \mathcal{H})$ (Here for any C*-algebra $B$ we denote by $Q(B)$ the quotient $M(B)/B$, where $M(B)$ is the multiplier algebra of $B$.) Conversely, to any *-homomorphism $C \to Q(A \otimes \mathcal{H})$ one can associate an extension by taking the pull-back of the fundamental extension

$$A \otimes \mathcal{H} \to M(A \otimes \mathcal{H}) \to Q(A \otimes \mathcal{H})$$

In E-theory to any extension $A \to B \to C$ one associates an asymptotic homomorphism $SC \to A$, and conversely, to any asymptotic homomorphism one can associate an extension by taking the pull-back

$$SA \to C^b(A) \to A_{\infty}$$

Here $C^b(A) = \{ f \in C^b : f(1) = 0 \}$. Note that here we get actually a suspension.

So it appears that what should be relevant for E-theory are (one usually works with stabilizations) extensions like

$$SA \otimes \mathcal{H} \to C^b(A \otimes \mathcal{H}) \to (A \otimes \mathcal{H})_{\infty}.$$

We shall think of this extension as an analog of the extension

$$SA \otimes \mathcal{H} \to M(SA \otimes \mathcal{H}) \to Q(SA \otimes \mathcal{H}).$$

Our result says that, exactly as for the multiplier algebras like $M(B \otimes \mathcal{H})$, the $K$-groups of algebras like $C^b(B \otimes \mathcal{H})$ are trivial. It can be easily shown that this statement is equivalent to the Theorem. Moreover, for $K_0$-groups the result holds even for nonstable algebras, that is, $K_0(C^b B) = 0$ for any C*-algebra $B$. Instead, for nonstable C*-algebras $B$ the group $K_1(C^b B)$ may not be trivial.

Finally we shall see that $\iota : A \to A_{\infty}$ induces, after suspension, an inverse for the connecting homomorphism $\partial : K_*(A_{\infty}) \to K_{*+1}(SA) = K_*(A)$.

We begin by examining the exactness properties for the functors $C^b : A \mapsto C^b(A)$ and $C^b_0 : A \mapsto C^b_0(A)$. 
Proposition. The correspondences $C^b$ and $C_0^b$ are exact functors.

Proof. Let $0 \to J \to A \to B \to 0$ be an exact sequence of $C^*$-algebras. The sequences

$$0 \to C^b J \to C^b A \to C^b B \to 0$$

are obviously exact at $A$ and $J$. The only problem can occur at $B$. Let $p : B \to A$ be a continuous map with $p(\lambda b) = \lambda p(b)$ for all $\lambda \in \mathbb{C}$, $b \in B$, such that $\pi \circ p = \text{Id}_B$. (Here we used, of course, the Bartle-Graves Theorem.) The homogeneity and continuity of $p$ give the existence of a constant $M > 0$ such that $\|p(b)\| \leq M\|b\|$ for all $b \in B$. So we can define the map $\tilde{p} : C^b B \to C^b A$ by $\tilde{p}(f) = p \circ f$, which gives a section for $C^b \pi$ and for $C_0^b \pi$ as well. This proves the surjectivity of the maps $C^b \pi$ and $C_0^b \pi$. □

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The functor $C^b$ allows one to define a new functor $A \mapsto A_\infty$ in the following way. For $\psi : A \to B$ a $*$-homomorphism one defines $\psi_\infty : A_\infty \to B_\infty$ as the unique $*$-homomorphism that makes the diagram

$$
\begin{array}{ccc}
C^b A & \xrightarrow{\pi} & A_\infty \\
\downarrow C^b \psi & & \downarrow \psi_\infty \\
C^b B & \xrightarrow{\pi_\infty} & B_\infty
\end{array}
$$

commutative. Since $(C^b \psi)(CA) \subset CB$, everything is correctly defined.

Lemma. If $\psi : A \to B$ is injective (resp. surjective) then so is $\psi_\infty : A_\infty \to B_\infty$.

Proof. Suppose $\psi$ is injective. Take $x \in A_\infty$ such that $\psi_\infty(x) = 0$. If we take $f \in C^b A$ with $x = \pi_A(f)$, we get $\pi_B(\psi \circ f) = 0$, that is, $\psi \circ f \in CB$. This reads $\lim_{t \to \infty} (\psi \circ f)(t) = 0$. But the injectivity of $\psi$ yields $\|(\psi \circ f)(t)\| = \|f(t)\|$, which gives $\lim_{t \to \infty} f(t) = 0$. That is, $f \in CA$, so $x = 0$. This shows the injectivity of $\psi_\infty$.

If $\psi$ is surjective, the surjectivity of $\psi_\infty$ follows from Proposition 1 since $C^b \psi$ is surjective. □

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The above property is used to prove the following.

Proposition. The functor $A \mapsto A_\infty$ is exact.

Proof. This is a standard fact which follows from the exactness of the functors $A \mapsto C^b A$, $A \mapsto C_0^b A$. □

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Next we will rephrase the statement of the Theorem using the canonical embedding $A \to C^b A$. To be more precise, let us define $\gamma_A : A \to C^b A$ by
\( \gamma_A(a)(t) = a \) for all \( a \in A, \ t \in [1, \infty) \). Define also \( \sigma_A : C^b_A \rightarrow A \) by \( \sigma_A(f) = f(1) \). The statement we are going to prove is:

**Theorem.** (i) For any C*-algebra \( A \) the map \( (\gamma_A)_* : K_0(A) \rightarrow K_0(C^b_A) \) is an isomorphism.

(ii) If \( A \) is stable, the the map \( (\gamma_A)_* : K_1(A) \rightarrow K_1(C^b_A) \) is also an isomorphism.

The first step in proving this result is the following fact. Note that \( \sigma \circ \gamma = \text{Id}_A \). This proves the injectivity part of the Theorem, that is: For any C*-algebra \( A \) the map \( (\gamma_A)_* : K_0(A) \rightarrow K_0(C^b_A) \) is injective.

So, what remains to be shown is the surjectivity.

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Before we go on with the proof, let us remark first that the stability assumption made in part (ii) of Theorem 4 is essential. To see this we examine the following (trivial) example. Take \( A = \mathbb{C} \), and let \( u \in C^b \mathbb{C} \) be the unitary given as \( u(t) = e^{it}, \ t \in [1, \infty) \). The first thing one observes is that \( u \) cannot be connected to 1 by a path of invertible elements in \( C^b \mathbb{C} \). Indeed, if this were the case it would follow that \( u \) can be written as a product of exponentials. Since we work with commutative algebras, this would give an element \( f \in C^b \mathbb{C} \) with \( u = e^f \). But this is clearly impossible. Let us turn our attention now to the class \([u]\) of \( u \) in \( K_1(C^b \mathbb{C}) \). If \([u] = 0\), this means that there exists a path \((U_s)_{s \in [0,1]}\) in some \( \text{GL}_n(C^b \mathbb{C}) \) with

\[
U_0 = \begin{pmatrix} u & \cdots \\ 1 & \cdots \\ \vdots & \nn & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & \cdots \\ 1 & \cdots \\ \vdots & \nn & 1 \end{pmatrix}.
\]

But if we take \( v_s = \det(U_s) \in C^b \mathbb{C} \), we again get a path of invertibles in \( C^b \mathbb{C} \) that connects \( u \) to 1, contradicting the previous remark. This proves that \([u] \neq 0\), so \( K_1(C^b \mathbb{C}) \neq \{0\} \).

6. Proof of Theorem 4, part (i)

Suppose first that \( A \) is unital. Let \( \alpha \in K_0(C^b A) \). Since \( \text{Mat}_n(C^b A) \simeq C^b(\text{Mat}_n(A)) \), we can suppose \( \alpha = [P] \) with \( P \) a projection in \( C^b A \). This means that \( P \) is a continuous function \( P : [1, \infty) \rightarrow A \) such that \( P(t) \) is a projection for all \( t \in [1, \infty) \).

Using Lemma 3.8. from [EK], there is a continuous function \([1, \infty) \ni t \mapsto V(t) \in A \) such that \( V(t) \) is a partial isometry and \( P(t) = V(t)^*V(t) \) for all \( t \). So if we take \( p = P(1) \in A \), this shows that \( P \) is Murray-von Neumann equivalent to \( \gamma(p) \), so \( \alpha \in \text{Ran} \gamma \).

In the nonunital case take \( \tilde{A} \) the algebra obtained from \( A \) by adjoining the unit. Take \( \epsilon : \tilde{A} \rightarrow \mathbb{C} \) as the canonical map and \( \nu : \mathbb{C} \rightarrow \tilde{A} \) as the obvious section for \( \epsilon \). Applying the \( C^b \) functor to the split exact sequence \( A \rightarrow \tilde{A} \xrightarrow{\epsilon} \mathbb{C} \)

we get a split exact sequence \( C^b A \rightarrow C^b(\tilde{A}) \overset{\epsilon_{\tilde{A}}}{\rightarrow} C^b \mathbb{C} \).
But then if we connect the corresponding sequences for $K_0$-groups we get a commutative diagram like

\[
\begin{array}{ccc}
K_0(C^bA) & \rightarrow & K_0(C^b(A)) \\
\uparrow (\gamma_A) & & \uparrow (\gamma_A) \\
K_0(A) & \rightarrow & K_0(A) \\
\end{array}
\Rightarrow
\begin{array}{ccc}
K_0(C^bC) & \rightarrow & K_0(C^b(A)) \\
\uparrow (\gamma_C) & & \uparrow (\gamma_C) \\
K_0(C) & \rightarrow & K_0(C) \\
\end{array}
\]

which gives the desired result. $\Box$

To prove part (ii) of Theorem 4, one cannot expect to use a similar argument for invertibles. The simple case $A = C$ shows that that stability condition is necessary. The result which we shall use is the following.

**Lemma.** If $A$ is stable, then $K_1(C^b(M(A))) = \{0\}$.

**Proof.** Let $\alpha \in K_1(C^b(M(A)))$. Since $\text{Mat}_n(M(A)) \cong M(\text{Mat}_n(A)) \cong M(A)$, we can assume $\alpha = [u]$ for some unitary $u \in C^b(M(A))$. That is, $u$ is a unitary map $u : [1, \infty) \rightarrow M(A)$.

Take then, for all $t \in [1, \infty)$, the unitary

\[
V(t) = \begin{pmatrix} u(t) & u(t)^* \\ u(t)^* & u(t) \\ \end{pmatrix} \in M(A).
\]

Clearly $V \in C^b(M(A))$. With the usual "rotation trick" $V$ can be connected to both

\[
\begin{pmatrix} u & 1 \\ 1 & \ddots \\ \end{pmatrix}
\] and \( \begin{pmatrix} 1 & 1 \\ 1 & \ddots \\ \end{pmatrix} \).

That is $[u] = [V] = 0$. $\Box$

**8. Proof of Theorem 4, part (ii)**

We shall reduce the proof to a "$K_0$-situation" for which we can apply the first part of the Theorem.

Suppose $A$ is stable. Take the exact sequences $C^bA \rightarrow C^b(M(A)) \rightarrow C^b(Q(A))$ and $A \rightarrow M(A) \rightarrow Q(A)$, and connect them through the corresponding $\gamma$-maps.

By the above Lemma the homomorphism $\partial : K_0(C^b(Q(A)) \rightarrow K_1(C^bA)$ is surjective. The same is true for the connecting homomorphism $\partial : K_0(Q(A)) \rightarrow K_1(A)$ (here we use the fact that $K_*(M(A)) = \{0\}$). But by naturality we have a commutative diagram

\[
\begin{array}{ccc}
K_0(C^b(Q(A)) & \rightarrow & K_1(C^bA) \\
\downarrow (\gamma_{Q(A)}) & & \downarrow (\gamma_A) \\
K_0(Q(A)) & \rightarrow & K_1(A) \\
\end{array}
\]

in which three arrows are surjective. Clearly this enforces the surjectivity of $(\gamma_A)$$. $\Box$
We turn our attention now to the original statement of the Theorem. In fact, everything follows from:

**Corollary.** (i) For any C*-algebra $A$ we have $K_0(C^b_b A) = \{0\}$.
(ii) For any stable C*-algebra $A$ we have $K_1(C^b_0 A) = \{0\}$.

Indeed, from the Theorem it follows that $\sigma_A : C^b A \to A$ induces an isomorphism between the $K$-groups. But $C^b_0 A = \text{Ker} \sigma_A$, which according to the exact sequence of $K$-theory, gives the desired result.

Let us regard $CA$ as the algebra of functions $f : [1, \infty) \to A$ with $f(1) = 0$ and such that $\lim_{t \to \infty} f(t)$ exists. If we consider the obvious embedding $CA \to C^b_0 A$ given by this description, then we have two exact sequences connected as follows:

\[
\begin{array}{ccc}
SA & \to & C^b_0 A \\
\downarrow & & \downarrow \uparrow i \\
SA & \to & CA \\
\uparrow & & \uparrow t \\
& & A
\end{array}
\]

By naturality, using the above Corollary we get that $i : K_*(A) \to K_*(A_{\infty})$ is an isomorphism.

Finally, if we take a “piece” of the exact sequence of $K$-groups we see that we have a commutative diagram like

\[
\begin{array}{ccc}
K_*(A_{\infty}) & \xrightarrow{\partial} & K_{*+1}(SA) \\
\downarrow t_* & & \downarrow \uparrow \partial \\
K_*(A) & \xrightarrow{\partial} & K_{*+1}(SA)
\end{array}
\]

This shows precisely, that modulo suspension, $i$ gives an inverse for $\partial : K_*(A_{\infty}) \to K_{*+1}(SA)$.

**Remark.** Lemma 7 has another alternative proof, which gives a stronger result. This is:

**Lemma.** Let $A$ be a stable C*-algebra and let $B = S(C^b(M(A)))$. Then the embedding $B \to \text{Mat}_2(B)$ given by $x \mapsto x \otimes e_{11}$ is homotopic to the null map.

**Proof.** Because of the isomorphism $M(A) \simeq \text{Mat}_2(M(A))$, we can view the above embedding as the map described in the following way. If we consider an element $x \in B$ as a function $f : (0, 1) \times [1, \infty) \to M(A)$, then our embedding sends $x$ to the function $g : (0, 1) \times [1, \infty) \to M(A)$ given by $g(s, t) = (f(s, t) 0 0)$. Take $\eta : B \to B$ to be the map given by $\eta(f)(s, t) = f(1 - s, t)$. Then our embedding is homotopic to

\[f \mapsto \begin{pmatrix} f & \eta(f) \\ \eta(f) & f \end{pmatrix},\]

which is clearly homotopic to the null map. (What we use here is that $\text{Id}_B \oplus \eta$ and $\eta \oplus \text{Id}_B$ are null homotopic.) □

**Comments.** In the above proof, as well as in the proof of Lemma 7, we have used the following fact: *If $A$ is stable, then one has a (canonical) $*$-homomorphism*
\[ \kappa : M(M(A) \otimes \mathcal{H}) \to M(A), \] and all the computations we have made took place in \( \text{Ran} \kappa \).

Another way is to use (see, for example, [Mi], Proposition 2.1) a sequence of orthogonal projections inside \( M(A) \) each of them equivalent to 1 and summing up to 1 (in the strict topology).

**References**


