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A CONVERSE OF THE VOLUME-MEAN-VALUE PROPERTY FOR INVARIANTLY HARMONIC FUNCTIONS

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ABSTRACT. It is shown that the balls are the only domains having the mean value property with respect to the invariantly harmonic functions in the unit ball of $\mathbb{C}^n$.

1. INTRODUCTION

Let B be the unit ball in $\mathbb{C}^n$. The Laplacian with respect to the Bergman metric in B is given in coordinates by

$$\Delta = (1-|z|^2)\sum_{i,j=1}^{n}(\delta_{ij} - z_i\bar{z}_j)D_i\bar{D}_j.$$

It is also called the *invariant Laplacian* because for every $f \in C^2(B)$ and every complex automorphism Ψ of B one has $\Delta(f\circ\Psi) = (\Delta f)\circ\Psi$. The group $\mathrm{Aut}(B)$ of such automorphisms is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a permuting a and 0. The functions f such that $\Delta f = 0$ are called $\mathcal{M}$-harmonic (we refer to [7, Chapter 4] for general properties of these functions).

If f is $\mathcal{M}$-harmonic in B, the mean-value of f on a sphere centered at 0 is $f(0)$ [7, p. 51]. Therefore, if B_r denotes the Euclidean ball of radius r centered at 0 and dm denotes normalized Lebesgue measure in B, for a measure $d\mu$ of type $d\mu = \varphi(|\zeta|)dm(\zeta)$ one has

$$f(0) = \frac{1}{\mu(B_r)}\int_{B_r} f\,d\mu$$

for all f $\mathcal{M}$-harmonic in B (in fact, for all f $\mathcal{M}$-harmonic and $d\mu$-integrable in B_r).

This is the case for the measure $d\lambda = (1-|\zeta|^2)^{-n-1}dm$. The measure $d\lambda$ is invariant under the action of $\mathrm{Aut}(B)$, whence

$$(*)\qquad f(a) = \frac{1}{\lambda(B_r)}\int_{\varphi_a(B_r)} f(\zeta)\,d\lambda(\zeta)$$

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for all f $\mathscr{M}$-harmonic and dm-integrable in $\varphi_a(B_r)$; i.e., f has the mean value property with respect to $d\lambda$ and the "Bergman balls" $\varphi_a(B_r)$. In this paper we prove a sort of converse of this fact:

Theorem 1. *Let U be open, connected, relatively compact in B, and such that $\partial U = \partial \bar{U}$. Assume $a \in U$ is such that*

$$f(a) = \frac{1}{\lambda(U)} \int_U f(\zeta)\,d\lambda(\zeta)$$

for all $\mathscr{M}$-harmonic functions f in a neighbourhood of $\bar{U}$. Then U is a Bergman ball centered at a; i.e., $U = \varphi_a(B_r)$ for some $r < 1$.

The analogue of this theorem for harmonic functions in Euclidean space (and Euclidean balls) was proved in [4] (see also [5] and [2]). See [1] for a converse, in another sense, of the mean-value property $(*)$.

2. The proof

We need some known facts about $\mathscr{M}$-harmonic functions that we will proceed to recall. There is a Riesz-type decomposition formula, valid at least for $u \in C^2(\bar{B})$,

$$u(z) = \int_S P(\zeta, z)u(\zeta)\,d\sigma(\zeta) + \int_B \Delta u(\zeta) G(\zeta, z)\,d\lambda(\zeta), \qquad z \in B.$$

Here $S = \partial B$, $d\sigma$ is the normalized Lebesgue measure on S, $P(\zeta, z)$ is the invariant (or Poisson-Szegö) kernel, and $G(\zeta, z)$ is the Green function with pole at z,

$$G(\zeta, z) = g(|\varphi_z(\zeta)|^2), \qquad g(x) = -c_n \int_x^1 \frac{(1-t)^{n-1}}{t^n}\,dt,$$

c_n being a positive constant. Moreover,

$$1 - |\varphi_z(\zeta)|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{\zeta} z|^2},$$

so G is symmetric. From the formula it follows that if f has compact support and is $\mathscr{M}$-harmonic in $\bar{U}$, then

$$f(z) = \int_{B \setminus \bar{U}} \Delta f(\zeta) G(\zeta, z)\,d\lambda(\zeta), \qquad z \in U.$$

This means that the functions $G_z(\zeta) = G(\zeta, z)$, $z \notin \bar{U}$, span the space of all $\mathscr{M}$-harmonic functions in a neighbourhood of $\bar{U}$. It also follows from the formula that under suitable conditions on Ψ, the Green potential

$$G\Psi(z) = \int_B \Psi(\zeta) G(\zeta, z)\,d\lambda(\zeta)$$

satisfies $\Delta G\Psi = \Psi$ in the weak sense.

Let us begin the proof of the theorem. Obviously we can assume $a = 0$. By what has been said, the hypothesis is equivalent to

$$cG(0, z) = \int_U G(\zeta, z)\,d\lambda(\zeta), \qquad z \notin \bar{U},$$

with $c = \lambda(U)$. Let us look at the potential

$$h(z) = \int_U G(\zeta, z)\, d\lambda(\zeta), \qquad z \in B.$$

We know that $h(z) = cG(0, z)$ for $z \notin \bar{U}$. In U, h satisfies $\Delta h = 1$; a computation shows that $\frac{1}{n}\log(1 - |z|^2)$ has invariant Laplacian 1 in B, hence

$$h(z) = \frac{1}{n}\log\frac{1}{1 - |z|^2} + u(z), \qquad z \in U,$$

with u $\mathscr{M}$-harmonic in U.

Lemma 1. *h is of class C^1 in B.*

Proof. One must prove that $\nabla_z G(\zeta, z)$ is locally uniformly integrable. Using the formulas above,

$$\begin{aligned}|D_i G(\zeta, z)| &= c_n \frac{(1 - |\varphi_z(\zeta)|^2)^{n-1}}{|\varphi_z(\zeta)|^{2n}} \frac{1 - |\zeta|^2}{|1 - \zeta\bar{z}||1 - \bar{\zeta}z|^2} \\ &\quad \times |\bar{\zeta}_i(1 - |z|^2) - \bar{z}_i(1 - \bar{\zeta}z)| \\ &= c_n \frac{(1 - |z|^2)^{n-1}(1 - |\zeta|^2)^{n-1}}{[|1 - \bar{\zeta}z|^2 - (1 - |z|^2)(1 - |\zeta|^2)]^n} \frac{1 - |\zeta|^2}{|1 - \zeta\bar{z}|} \\ &\quad \times |\bar{\zeta}_i(1 - |z|^2) - \bar{z}_i(1 - \bar{\zeta}z)|.\end{aligned}$$

Therefore,

$$|\nabla_z G(\zeta, z) \le C \frac{|\zeta - z|}{[|1 - \bar{\zeta}z|^2 - (1 - |z|^2)(1 - |\zeta|^2)]^n}, \qquad \zeta, z \in B.$$

This is a well-known singularity; in fact, it is the same that appears in the Cauchy kernel for the ball. If $e_1, \dots, e_{n-1}$ is an orthonormal basis of $(\mathbb{C}_z)^\perp$ and $\zeta - z = \lambda_n z + \sum \lambda_i e_i$, the expression inside the brackets equals

$$|\lambda_n|^2|z|^2 + (1 - |z|^2)\sum_{i=1}^{n-1} |\lambda_i|^2 \ge (1 - |z|^2)|\zeta - z|^2.$$

Hence, for z in a compact set $K \subset B$,

$$|\nabla_z G(\zeta, z)| \le C_K |\zeta - z|^{1-2n},$$

which proves the lemma. □

By Lemma 1, $u \in C^1(\bar{U})$, and since $\partial U = \partial\bar{U}$, it follows that u coincides on ∂U with

$$cG(0, z) - \frac{1}{n}\log\frac{1}{1 - |z|^2},$$

a radial function, up to order one.

Our objective is to prove that u is constant. At this point, we need to introduce the tangent fields

$$R = \sum_1^n z_j D_j = N + iT, \; L_{jk} = \bar{z}_j D_k - \bar{z}_k D_j, \qquad j \neq k.$$

The fields T, $\operatorname{Re} L_{jk}$, and $\operatorname{Im} L_{jk}$ span at each point z the tangent space to the sphere of radius $|z|$. We write

$$\Delta_0 = \sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) D_i \bar{D}_j .$$

By direct computation one proves

Lemma 2. (a) $\Delta_0 T - T\Delta_0 = 0$.
(b) *If L denotes one of the L_{jk}, $\Delta_0 L - L\Delta_0 = 2iLT$.*

By Lemma 2(a), Tu is also $\mathscr{M}$-harmonic in U. But u is radial on ∂U up to order 1; hence, $Tu = 0$ on ∂U. By the maximum principle for $\mathscr{M}$-harmonic functions, $Tu \equiv 0$ in U.

Lemma 2(b) implies then that each $L_{jk}u$ is also $\mathscr{M}$-harmonic in U. As u is radial on ∂U up to order 1, it follows as before that $L_{jk}u \equiv 0$ in U for all j, k.

We have thus proved that

$$Tu = L_{jk}u = 0 \quad \text{in } U .$$

Let B' be a ball centered at 0 included in U; then u is radial in B'. Being $\mathscr{M}$-harmonic in B' and radial, it must be constant. By analytic continuation, u is constant, say k, in U.

Then

$$cg(|z|^2) - \frac{1}{n} \log \frac{1}{1 - |z|^2} = k , \qquad z \in \partial U.$$

The left-hand side, as a function of $|z|$, increases from $-\infty$ to a maximum value and then decreases again to $-\infty$ when $|z| \to 1$. Therefore, $|z|$ takes at most two values on ∂U, and since U was assumed to contain 0, U must be a ball centered at 0.

3. Remarks and questions

(a) The following approximation theorem can be obtained by combining a general result of [6] on the Runge property for functions annilated by analytic-hypoelliptic operators and [7, 5.5.4] (see also [3] for a different proof).

Theorem 2. *Assume K is a compact set in B such that $B \setminus K$ is connected. Then every $\mathscr{M}$-harmonic function f in a neighbourhood of K is the uniform limit on K of a sequence of $\mathscr{M}$-harmonic functions in the whole ball B and continuous on $\bar{B}$.*

Hence if U is as in the theorem and, moreover, $B \setminus \bar{U}$ is connected, then from the assumption

$$f(a) = \frac{1}{\lambda(U)} \int_U f \, d\lambda$$

for all f $\mathscr{M}$-harmonic in B and continuous on $\bar{B}$ it follows that U is a Bergman ball centered at a. We do not know if this holds true when $B \setminus \bar{U}$ is not connected.

In the case of harmonic functions on Euclidean space, Shapiro [8] provided us an example of a (doubly connected) open set A of $\mathbb{R}^2$ such that $0 \in A$, $\int_A z^n\, dm(z) = 0 \ \forall n \geq 1$, and A is not a disc centered at 0.

(b) The same method proves the following result: let U be as in the theorem, $0 \in U$, and let $\varphi(t)$, $0 \leq t < 1$, be a continuous strictly positive function. Assume that

$$f(0) = \frac{1}{c} \int_U f(\zeta)\varphi(|\zeta|^2)\, dm(\zeta)$$

with $c = \int_U \varphi(|\zeta|^2)\, dm(\zeta)$ for all $\mathcal{M}$-harmonic functions in $\bar{U}$. Then U is a ball centered at 0. The only difference is that $\frac{1}{n} \log 1/(1 - |z|^2)$ must be replaced by $\Psi(|z|^2)$, the unique radial function vanishing at zero whose invariant Laplacian is $\varphi(|z|^2)(1 - |z|^2)^{n+1}$. It is easy to check that

$$\Psi'(x) = h(x)g'(x),$$

where $h(x) = \int_0^x \varphi(t)t^{n-1}\, dt$. Then the equation $cg - \Psi = k$ has again at most two solutions, because $(cg - \Psi)' = (c - h)g'$ changes sign only once, h being strictly increasing.

(c) In dimension $n = 1$, the invariant Laplacian is $(1 - |z|^2)D\bar{D}$; hence, the $\mathcal{M}$-harmonic functions are the usual harmonic functions in the plane. A Euclidean disk therefore has the mean-value property, with respect to the Lebesgue measure. In dimension $n > 1$, by remark (b), the balls are the only domains having the property

$$cf(0) = \int_U f(\zeta)\, dm(\zeta), \qquad f\ \mathcal{M}\text{-harmonic in } \bar{U},\ 0 \in U.$$

The question arises whether this holds true for another point: what pairs (a, U) $a \in U$ have the property

$$cf(a) = \int_U f(\zeta)\, dm(\zeta), \qquad c = m(U),\ f\ \mathcal{M}\text{-harmonic in } \bar{U}?$$

Does it follow that $a = 0$ (and so U is a ball)?

(d) The theorem in the case of harmonic function in $\mathbb{R}^n$ is still true if U is of finite area but not necessarily bounded. In our case, is our theorem still true if U is no more relatively compact in B?

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