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A NOTE ON ENDOMORPHISMS OF IRRATIONAL ROTATION C*-ALGEBRAS

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ABSTRACT. Let θ be an irrational number and A_θ the corresponding irrational rotation C*-algebra. In the present note we will show that proper endomorphisms of A_θ of a certain sort do not occur if θ is not quadratic.

Let θ be an irrational number in $[0, 1]$ and A_θ the corresponding irrational rotation C*-algebra. Let τ be the unique tracial state on A_θ. In the present note we will consider endomorphisms Φ of A_θ satisfying the following conditions:

(1) There is a conditional expectation of index-finite type of A_θ onto $\Phi(A_\theta)$,

(2) $\Phi(A_\theta)' \cap A_\theta \neq \mathbf{C}1$,

where we say endomorphisms when we mean proper unital *-endomorphisms.

Theorem. *If θ is not quadratic, there is no endomorphism of A_θ satisfying conditions* (1) *and* (2).

Proof. We suppose that there is an endomorphism Φ of A_θ satisfying conditions (1) and (2). Then by Watatani [6, Proposition 2.7.3] $\Phi(A_\theta)' \cap A_\theta$ is finite dimensional. Let p be a minimal projection in $\Phi(A_\theta)' \cap A_\theta$. Then there are integers k, l such that $0 < \tau(p) = k + l\theta \leq 1$. Let Ψ be a unital monomorphism of A_θ to $pA_\theta p$ defined for any $x \in A_\theta$ by $\Psi(x) = p\Phi(x)$. Let τ_1 be a tracial state on A_θ defined for any $x \in A_\theta$ by

$$\tau_1(x) = \frac{1}{k + l\theta}\tau \circ \Psi(x).$$

By the uniqueness of the tracial state on A_θ, we can see that $\tau_1 = \tau$. Let q be a projection in A_θ with $\tau(q) = \theta$. Then $\tau_1(q) = \frac{1}{k+l\theta}\tau(\Psi(q))$. Since $\Psi(q) \in pA_\theta p \subset A_\theta$, there are integers s, t such that $\tau(\Psi(q)) = s + t\theta$. Thus $\tau_1(q) = \frac{s+t\theta}{k+l\theta}$. Hence we obtain that $\theta = \frac{s+t\theta}{k+l\theta}$. Thus $l\theta^2 + (k - t)\theta - s = 0$. Since θ is not quadratic, $l = 0$. Therefore, $\tau(p) = k \in \mathbf{Z}$. Since $0 < \tau(p) \leq 1$ and τ is faithful, $p = 1$. Hence $\Phi(A_\theta)' \cap A_\theta = \mathbf{C}1$. This is a contradiction. Q.E.D.

We suppose that θ is a quadratic irrational number such that there are integers s and $t < 0$ such that $t^2 - Ds^2 = -4$ where D is the discriminant of θ.

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Then there is an endomorphism of A_θ satisfying conditions (1) and (2). We will show that we can construct one in the same way as in [1]:

In the same way as in [1, Lemmas 1–3] we can see that there is a nontrivial fractional transformation

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbf{Z})$$

such that $0 < a + b\theta < 1$, $a + d < 0$, and

$$\theta = \frac{c + d\theta}{a + b\theta},$$

where $\mathrm{GL}(2, \mathbf{Z})$ is the group of all (2×2)-matrices over $\mathbf{Z}$ with determinant ± 1. Let $n = -t$ and $r = a + b\theta$. Then we obtain that $r^2 + nr = 1$ since $n^2 - Ds^2 = -4$. Furthermore in the same way as in [1, Lemmas 7 and 8] we can find an orthogonal family $\{p_j\}_{j=1}^{n+1}$ of projections in A_θ such that $\tau(p_j) = r$ for $j = 1, 2, \ldots, n$ and $\tau(p_{n+1}) = r^2$ since $r^2 + nr = 1$. Hence by Rieffel [5, Theorem1.1 and Corollary 2.6] $p_j A_\theta p_j \cong A_\theta$ for $j = 1, 2, \ldots, n+1$. Let ϕ_j be an isomorphism of A_θ onto $p_j A_\theta p_j$ for $j = 1, 2, \ldots, n+1$ and Φ an endomorphism of A_θ defined by $\Phi = \sum_{j=1}^{n+1} \phi_j$. Then in the same way as in [1, §3] Φ satisfies condition (1), and by its definition it satisfies condition (2). Therefore, we obtain an endomorphism Φ of A_θ as desired.

Remark 1. Let θ be a quadratic irrational number with its discriminant D. Then the following conditions are equivalent:

(A) There are integers s and $t < 0$ such that $t^2 - Ds^2 = -4$,
(B) θ has an odd period of its continued fraction expansion.

Remark 2. The author does not know whether there is an endomorphism of A_θ satisfying conditions (1) and (2) when θ is a quadratic irrational number not satisfying condition (A).

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