PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 4, December 1994

ON UNIQUENESS SETS FOR AREALLY MEAN p-VALENT FUNCTIONS

ENRIQUE VILLAMOR

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. We study the sets of uniqueness of areally mean p-valent functions in the unit disc. Namely, if $f(z)$ is in this class and has the same angular limit in a set E on the boundary of the unit disc, we prove that if p is small compared to the size of E then $f(z)$ is constant. We then construct an areally mean p-valent function which shows that some condition on the size of the set E must be imposed.

INTRODUCTION

The original F. and M. Riesz Theorem states that if a bounded analytic function in the unit disc Δ has the same radial limit in a set of positive Lebesgue measure E in $\partial\Delta$ then the function has to be constant.

Beurling [1, 3] showed that if we consider the class of univalent functions in the unit disc, the same result holds if we replace a set of positive Lebesgue measure by a set of positive logarithmic capacity in $\partial\Delta$.

We start by giving the definition of areally mean p-valent functions.

Definition 1. Let $f(x)$ be a regular nonconstant function in Δ. Define

$$n(w) = n(w, \Delta, f)$$

to be the number of roots of the equation $f(x) = w$ in Δ, and write

$$p(R) = p(R, \Delta, f) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\theta})\, d\theta.$$

Then if there exists a positive number p such that

$$\int_0^R p(\rho)2\rho\, d\rho \le pR^2$$

for all positive R, we say that the function $f(z)$ is an areally mean p-valent function.

From now on we are going to denote this class of functions by AMP. This class has been studied by several authors; good references are Hayman [5] and Eke [4].

Let us consider now the following class of functions.

Received by the editors December 3, 1990 and, in revised form, March 18, 1993.
1991 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Areally mean p-valent, logarithmic capacity.

Definition 2. Let $f(x)$ be a regular function in the unit disc. If

$$\iint_{\Delta} \frac{|f'(z)|^2}{[1+|f(z)|^2]^2}\, dx\, dy < \infty,$$

we say that $f(z) \in D_S$. These functions are called functions of finite spherical area.

It is not difficult to show that $\mathrm{AMP} \subset D_S$. Beurling [1] proved the following theorem.

Theorem A. *Suppose that* $f(x) \in D_S$ *and that*

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \alpha,$$

whenever $e^{i\theta} \in E$, *where* $E \subset \partial\Delta$.

We define

$$\delta_\rho(\alpha) = \{w : |w-\alpha| < \rho\},$$
$$\Delta_\rho(\alpha) = f^{-1}(\delta_\rho(\alpha)) = \{z \in \Delta : |f(z)-\alpha| < \rho\},$$

and

$$\iint_{\Delta_\rho(\alpha)} \frac{|f'(z)|^2}{[1+|f(z)|^2]^2}\, dx\, dy = A_\rho(\alpha).$$

If now $\mathrm{cap}(E) > 0$ *and* $\limsup_{\rho\to 0}[\frac{A_\rho(\alpha)}{\rho^2}] < \infty$, *then* $f(z)$ *is constant.*

Later Tsuji [9] gave a modified version of this theorem.

Carleson [2] proved that some condition on the limiting value α must be imposed if we want to obtain a uniqueness result for the class D_S. He constructed a nonconstant function $f(z) \in D_S$ such that $\lim_{r\to 1} f(re^{i\theta}) = 0$ for all $e^{i\theta} \in E \subset \partial\Delta$, and $\mathrm{cap}(E) > 0$.

Functions of finite spherical area are only apparently more general than those in AMP. A function f belongs to D_S if and only if some bilinear transform $\frac{af+b}{cf+d}$ for suitable constants a, b, c, d belongs to AMP (possibly as a meromorphic function). For AMP 0 and ∞ are special points, while D_S is invariant under bilinear transforms.

Due to the above remarks and using Carleson's construction in [2], we shall construct a nonconstant function $f(z)$ in AMP for some p, such that $f(z)$ has the same angular limit in a set of positive capacity.

In the positive direction we will prove that if $f(z)$ in AMP has the same nontangential limit in a set E of positive capacity and if p is small compared to the size of E, then $f(z)$ is constant. We will have to make precise the above statement in Theorem 1.

We start with some preliminaries. Let $f(z)$ be in AMP, such that $\lim_{z\to e^{i\theta}} f(z) = \alpha$ nontangentially for all $e^{i\theta} \in E$ and $\mathrm{cap}(E) > 0$. Now we want to reduce the problem to the case in which $f(z)$ is zero free in some simply connected domain $\Omega \subset \Delta$. It is known [5] that any areally mean p-valent function can have at most p zeros counting multiplicity. Let z_j, $j = 1, \ldots, k$ be the points for which $f(z_j) = 0$. We define $r_0 = \max_{1\le j\le k} |z_j|$. Let Ω be the simply connected domain given by

$$\Omega = \{z : r_0 < |z_j| < 1,\ |\arg z| < \pi\}.$$

Then $f(z)$ is areally mean p-valent in Ω, $f(z) \neq 0$, and $f(z)$ has the same nontangential limit α on E, where $E \subset \partial\Omega$ and $\operatorname{cap}(E) > 0$.

Then for each positive integer n, $g(z) = f^{1/n}(z)$ is single valued and $\alpha^{1/n}$ might take n different values. We call these values $\alpha_{i,n}$, $i = 1, \dots, n$. Let $E_{i,n} \subset E$, $i = 1, \dots, n$, be the set such that $\lim_{z\to e^{i\theta}} g(z) = \alpha_{i,n}$ for any $e^{i\theta} \in E_{i,n}$. It is clear that $E = \bigcup_{i=1}^{n} E_{i,n}$ and the $E_{i,n}$ are disjoint. Since $\operatorname{cap}(E) > 0$, there exists at least one $i \in \{1, \dots, n\}$ such that $\operatorname{cap}(E_{i,n}) > 0$. Among those, we choose $E_{i_0,n}$ with the property that,

$$\operatorname{cap}(E_{i_0,n}) = \max_{1\leq i\leq n} \operatorname{cap}(E_{i,n}) > 0.$$

Let $\gamma(E_{i_0,n})$ be the Robin constant of the set $E_{i_0,n}$.

Let $f(z)$ in AMP be zero free, let $0 < \lambda < 1$, and recall that $2\pi p(R, f)$ is the total variation of $\arg f$ on the level curves $|f(z)| = R$; then $p(R^\lambda, f^\lambda) = \lambda p(R, f)$. We want to show that the function $f^\lambda(z)$ is areally mean $p\lambda$-valent. Thus, we have to show that,

$$\int_0^{R^\lambda} p(t, g)\, dt^2 \leq p\lambda R^{2\lambda}$$

for any positive R, or

$$\int_0^R p(s^\lambda, f^\lambda)\, d(s^{2\lambda}) \leq p\lambda R^{2\lambda},$$

or, which is the same,

$$\int_0^R \lambda^2 p(s, f) 2s^{2\lambda-1}\, ds \leq p\lambda R^{2\lambda}.$$

We write $W(R) = 2\int_0^R p(s, f) s\, ds$ so that, since $f \in \text{AMP}$, $W(R) \leq pR^2$. Then for $0 < \lambda < 1$

$$\begin{aligned}
\int_0^R p(s, f) 2s^{2\lambda-1}\, ds &= \int_0^R s^{2\lambda-2}\, dW(s) \\
&= R^{2\lambda-2} W(R) + (2-2\lambda) \int_0^R W(s) s^{2\lambda-3}\, ds \\
&\leq R^{2\lambda-2} W(R) + (2-2\lambda) \int_0^R p s^{2\lambda-1}\, ds \\
&\leq pR^{2\lambda} \left[1 + \frac{2-2\lambda}{2\lambda}\right] = \frac{p}{\lambda} R^{2\lambda}.
\end{aligned}$$

Multiplying by λ^2, we obtain that

$$\int_0^{R^\lambda} p(t, g)\, dt^2 \leq p\lambda R^{2\lambda}$$

for any positive R, as we wanted to show.

After these preliminaries we state our theorem.

Theorem 1. *Suppose that* $f(z) \in \text{AMP}$ *and that*

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \alpha$$

for any $e^{i\theta} \in E$, *where* $\operatorname{cap}(E) > 0$. *Then, if*

$$\liminf_{n\to\infty}\left[\frac{\gamma(E_{i_0,n})}{n}\right] < \frac{1}{4\pi^2 p},$$

the function $f(z)$ *is constant.*

The natural question to ask is how sharp is our theorem. Namely, for fixed p is it true that for any positive ε there exists a nonconstant function $f(z) \in \mathrm{AMP}$ such that

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \alpha$$

for every $e^{i\theta} \in E$, where $\operatorname{cap}(E) > 0$, and such that

$$\liminf_{n\to\infty}\left[\frac{\gamma(E_{i_0,n})}{n}\right] > \frac{1}{4\pi^2 p} - \varepsilon?$$

1. Proofs

Proof of Theorem 1. The case $\alpha = 0$ is trivial, since then for $f(z) \in \mathrm{AMP}$ the value $\alpha = 0$ will satisfy the hypotheses of Theorem A. The case $\alpha = \infty$ can be treated in the same way by considering the function $g(z) = \frac{1}{f(z)}$, which is in the class D_S. The function $g(z)$ satisfies the hypotheses of Theorem A for the value $\alpha = 0$. Therefore, we can assume that $\alpha \neq 0, \infty$.

Suppose that there exists a function $f(z)$ such that $f(z) \in \mathrm{AMP}$ and

$$\lim_{r\to 1} f(re^{i\theta}) = \alpha,$$

when $e^{i\theta} \in E$, where $\operatorname{cap}(E) > 0$.

By a result in [8], if $f(z) \in \mathrm{AMP}$, thus $f(z) \in D_S$, then $f(z)$ is normal. Hence by a theorem in [7], if $f(z)$ is normal, radial limits of $f(z)$ are also nontangential limits.

By the observation we made in the introduction, we can assume that $f(z)$ is areally mean p-valent in $\Omega = \{z : r_0 < |z| < 1,\ |\arg z| < \pi\}$, $f(z) \neq 0$, and $f(z)$ has the same nontangential limit α on E, where $E \subset \partial\Omega$ and $\operatorname{cap}(E) > 0$.

The function $g(z) = f^{1/n}(z)$ is areally mean $\frac{p}{n}$-valent. For fixed n, choose i_0 as in the introduction. We have that

$$\iint_{\Delta_\rho(\alpha_{i_0,n})} |g'(z)|^2\,dx\,dy \le \iint_{\Omega_\rho(\alpha_{i_0,n})} |g'(z)|^2\,dx\,dy \le \frac{p\pi}{n}[\rho + |\alpha|^{1/n}]^2,$$

where $\Delta_\rho(\alpha_{i_0,n}) = \{z \in \Omega : |g(z) - \alpha_{i_0,n}| < \rho\}$, since $\Delta_\rho(\alpha_{i_0,n}) \subset \{z \in \Omega : |g(z)| < \rho + |\alpha_{i_0,n}|\} = \Omega_\rho(\alpha_{i_0,n})$; observe that $|\alpha_{i_0,n}| = |\alpha|^{1/n}$.

Let τ be a small positive number to be determined later, which is going to depend only on the function $f(z)$. Considering $\tau^{1/n} = \rho$ in the above inequalities we obtain

$$\begin{aligned}\iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|^2\,dx\,dy &\le \iint_{\Omega_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|^2\,dx\,dy \\ &\le \frac{p\pi}{n}[\tau^{1/n} + |\alpha|^{1/n}]^2.\end{aligned} \tag{1.1}$$

Without loss of generality we can assume that the set $E_{i_0,n}$ is closed. Then there exists a distribution $\mu(\zeta)$ of total mass 1 on $E_{i_0,n}$ such that the potential

$$u(z) = \int_{E_{i_0,n}} \log\left|\frac{1}{z-\zeta}\right| d\mu(\zeta)$$

is bounded by $V_0(E_{i_0,n}) = \gamma(E_{i_0,n})$ for any z in the complex plane. Standard computations [3, pp. 58–59] show that

$$\iint_{|z|<1} \left[\frac{\partial u}{\partial r}\right]^2 r\,dr\,d\theta \le \frac{\pi}{2}\,[\gamma(E_{i_0,n})] < \infty. \tag{1.2}$$

Define

$$S_n = \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|\frac{\partial u}{\partial r} r\,dr\,d\theta.$$

By the Schwarz's inequality, (1.1) and (1.2)

$$S_n \le \pi\left[\frac{p(\gamma(E_{i_0,n}))}{2n}\right]^{1/2} [\tau^{1/n} + |\alpha|^{1/n}]. \tag{1.3}$$

Define now

$$\sigma_n(\zeta) = -\iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|\,d[\arg(re^{i\theta}-\zeta)]\,dr$$

for $z = re^{i\theta}$. The Cauchy-Riemann equations for the function $u(z)$ give us that

$$\frac{\partial u}{\partial r} r\,d\theta = -\int_{E_{i_0,n}} d[\arg(re^{i\theta}-\zeta)]\,d\mu(\zeta).$$

Therefore, we can write

$$S_n = \int_{E_{i_0,n}} \sigma_n(\zeta)\,d\mu(\zeta).$$

Our goal is to get an estimate of $\sigma_n(\zeta)$ from below for any $\zeta \in E_{i_0,n}$.

By the above remarks it is enough to estimate $\sigma_n(\zeta)$ at one point of $E_{i_0,n}$, since the same estimate will hold at any other point of $E_{i_0,n}$. We can assume that $\zeta = 1$ is in $E_{i_0,n}$, and hence we have to estimate

$$\sigma_n(1) = -\iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|\,d[\arg(re^{i\theta}-1)]\,dr.$$

For $-\frac{\pi}{2} < t < \frac{\pi}{2}$ we define l_t to be a rectilinear segment of length $\cos t$ lying in $|z| < 1$ and making an angle t at $\zeta = 1$ with the radius drawn to $\zeta = 1$. Call $\tilde{t} = -\arg(re^{i\theta}-1)$; then we have

$$\sigma_n(1) = \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|\,d\tilde{t}\,dr.$$

We know that $\lim_{z\to 1} g(z) = \alpha_{i_0,n}$ in any angular domain. Let $\omega = \bar{\omega} \cap \{z : |z-\frac{1}{2}| < \frac{1}{2}\}$ be the angular domain resulting of the intersection of an angular domain $\bar{\omega}$ which has its vertex at $\zeta = 1$ and is symmetrical to the radius of $|z| = 1$ through $\zeta = 1$ and is of aperture $\frac{\pi}{2}$, with the disc $\{z : z - \frac{1}{2}| < \frac{1}{2}\}$. Then the part of ω in the vicinity of $\zeta = 1$ belongs to $\Delta_{\tau^{1/n}}(\alpha_{i_0,n})$.

Let Δ denote the common part of $\Delta_{\tau^{1/n}}(\alpha_{i_0,n})$ and this angular domain ω. Observe that $\Delta_{\tau^{1/n}}(\alpha_{i_0,n}) = \{z \in \Omega : |f^{1/n}(z) - \alpha_{i_0,n}| < \tau^{1/n}\} \subset \{z \in \Omega : |f(z) - \alpha| < \tau\}$; therefore, for τ small enough depending only on the function $f(z)$, the connected component of Δ with the point $\zeta = 1$ as boundary point lies inside the circle $|z - \frac{3}{4}| = \frac{1}{4}$. Hence,

$$\sigma_n(1) = \iint_\Delta |g'(z)|\, d\tilde{t}\, dr + \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})\backslash\Delta} |g'(z)|\, d\tilde{t}\, dr = \mathrm{I} + \mathrm{II}.$$

Consider the range where $d\tilde{t} < 0$ in II, and call the corresponding integral III. Then

$$\sigma_n(1) = \mathrm{I} - |\mathrm{III}|.$$

By the definition of $\tilde{t}$

$$\frac{d\tilde{t}}{d\theta} = \frac{r\cos\theta - r^2}{1 + r^2 - 2r\cos\theta}.$$

Fix $r = \cos\theta_0$ so that the point $z = re^{i\theta}$ lies outside the circle $|z - \frac{1}{2}| = \frac{1}{2}$ (i.e., $r = \cos\theta$) whenever $\theta_0 \le |\theta| \le \pi$. We observe that $d\tilde{t} \ge 0$ for $|\theta| \le \theta_0$ and $d\tilde{t} \le 0$ for $\theta_0 \le |\theta| \le \pi$. It is not difficult to see that for $\theta_0 \le |\theta| \le \pi$ we have $|d\tilde{t}| \le r\, d\theta$. Now

$$|\mathrm{III}| \le \int dr \int_{\theta_0 \le |\theta| \le \pi} |g'(z)|\, |d\tilde{t}|,$$

where the integral is restricted to the region $\Delta_{\tau^{1/n}}(\alpha_{i_0,n})\backslash\Delta$. So as we know that for $\theta_0 \le |\theta| \le \pi$, $|d\tilde{t}| \le r\, d\theta$, by Schwarz's inequality,

$$\begin{aligned}
|\mathrm{III}| &\le \int dr \int_{\theta_0 \le |\theta| \le \pi} |g'(z)|\, |d\tilde{t}| \le \int dr \int_{\theta_0 \le |\theta| \le \pi} |g'(z)| r\, d\theta \\
&\le \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)| r\, dr\, d\theta \\
&\le \left\{ \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} |g'(z)|^2 r\, dr\, d\theta \right\}^{1/2} \left\{ \iint_{\Delta_{\tau^{1/n}}(\alpha_{i_0,n})} r\, dr\, d\theta \right\}^{1/2} \\
&\le \pi \left[\frac{p}{n} [\tau^{1/n} + |\alpha|^{1/n}]^2 \right]^{1/2}.
\end{aligned}$$

Therefore,

$$\sigma_n(1) \ge \mathrm{I} - |\mathrm{III}| \ge \iint_\Delta |g'(z)|\, d\tilde{t}\, dr - \pi \left[\frac{p}{n} [\tau^{1/n} + |\alpha|^{1/n}]^2 \right]^{1/2}.$$

Now we pass to estimate I from below, namely,

$$\mathrm{I} = \iint_\Delta |g'(z)|\, d\tilde{t}\, dr.$$

If we set $\tilde{\rho} e^{it} = 1 - re^{-i\theta}$, a calculation shows that

$$d\tilde{t}\, dr = \frac{\cos t - \tilde{\rho}}{(1 + \tilde{\rho}^2 - 2\tilde{\rho}\cos t)^{1/2}}\, d\tilde{\rho}\, dt.$$

Also, since $(1+\tilde{\rho}^2-2\tilde{\rho}\cos t)\le 1$ in $|z-\frac{1}{2}|\le\frac{1}{2}$, we have that

$$\sigma_n(1)\ge\int_{-\pi/4}^{\pi/4}dt\int_{l_t\cap\Delta}\frac{|g'(z)|(\cos t-\tilde{\rho})}{(1+\tilde{\rho}^2-2\tilde{\rho}\cos t)^{1/2}}\,d\tilde{\rho}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2}$$
$$\ge\int_{-\pi/4}^{\pi/4}dt\int_{l_t\cap\Delta}|g'(z)|(\cos t-\tilde{\rho})\,d\tilde{\rho}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2}.$$

If we consider now $l_t/2$ to be half of the segment l_t, the half having the point $\zeta=1$ as one of its end points, then $(\cos t-\tilde{\rho})\ge\frac{\cos t}{2}$ on $[l_t/2\cap\Delta]$ for each $\frac{-\pi}{4}\le t\le\frac{\pi}{4}$, and the other end point of $l_t/2$ lies on $|z-\frac{3}{4}|=\frac{1}{4}$. Thus,

$$\sigma_n(1)\ge\int_{-\pi/4}^{\pi/4}\frac{\cos t}{2}\,dt\int_{l_t\cap\Delta}|g'(z)|\,d\tilde{\rho}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2}.$$

By our construction $[l_t/2\cap\Delta]$ contains a segment joining the point $\zeta=1$ with a boundary point of $\Delta_{\tau^{1/n}}(\alpha_{i_0,n})$, since by our choice of τ the connected component of Δ with the point $\zeta=1$ as boundary point lies inside the circle $|z-\frac{3}{4}|=\frac{1}{4}$ and one of the end points of $l_t/2$ lies on $|z-\frac{3}{4}|=\frac{1}{4}$. Therefore,

$$\int_{l_t/2\cap\Delta}|g'(z)|\,d\tilde{\rho}\ge\tau^{1/n}$$

for each $-\frac{\pi}{4}\le t\le\frac{\pi}{4}$. Hence

$$\sigma_n(1)\ge\frac{\sqrt{2}}{2}\tau^{1/n}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2},$$

and the same estimate holds for each $\zeta\in E_{i_0,n}$; therefore,

$$S_n\ge\frac{\sqrt{2}}{2}\tau^{1/n}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2}. \tag{1.4}$$

By (1.3) and (1.4)

$$\pi\left[\frac{p(\gamma(E_{i_0,n}))}{2n}\right]^{1/2}[\tau^{1/n}+|\alpha|^{1/n}]\ge S_n\ge\frac{\sqrt{2}}{2}\tau^{1/n}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2},$$

for any $n>0$. Hence we must have

$$\pi\left[\frac{p(\gamma(E_{i_0,n}))}{2n}\right]^{1/2}[\tau^{1/n}+|\alpha|^{1/n}]\ge\frac{\sqrt{2}}{2}\tau^{1/n}-\pi\left[\frac{p}{n}[\tau^{1/n}+|\alpha|^{1/n}]^2\right]^{1/2}, \tag{1.5}$$

dividing both sides by $[\tau^{1/n}+|\alpha|^{1/n}]$, we have

$$\pi\left[\frac{p(\gamma(E_{i_0,n}))}{2n}\right]^{1/2}\ge\frac{\sqrt{2}}{2}\frac{\tau^{1/n}}{[\tau^{1/n}+|\alpha|^{1/n}]}-\pi\left[\frac{p}{n}\right]^{1/2},$$

squaring both sides, taking the $\liminf$ as $n\to\infty$, and dividing both sides by π^2p, we obtain

$$\liminf_{n\to\infty}\left[\frac{\gamma(E_{i_0,n})}{n}\right]\ge\frac{1}{4\pi^2p},$$

which proves our theorem.

As an immediate corollary to Theorem 1 we obtain an estimate in how big the size of E can be for the function $f(z)$ not to be necessarily a constant. More precisely,

Corollary 1. *Suppose that* $f(z) \in \mathrm{AMP}$ *and that*

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \alpha$$

for any $e^{i\theta} \in E$. *Then if* $\operatorname{cap}(E) > 2e^{-1/4\pi^2 p}$, *the function* $f(z)$ *is constant.*

Proof of Corollary 1. By Theorem 7.17 in [10, p. 437] we have

$$\frac{\log 2 + \gamma(E_{i_0,n})}{n} \le \log 2 + \gamma(E) = \log\left[\frac{2}{\operatorname{cap}(E)}\right].$$

By hypothesis,

$$\frac{1}{4\pi^2 p} > \log\left[\frac{2}{\operatorname{cap}(E)}\right];$$

hence,

$$\frac{\log 2 + \gamma(E_{i_0,n})}{n} < \log\left[\frac{2}{\operatorname{cap}(E)}\right] < \frac{1}{4\pi^2 p},$$

taking the $\liminf_{n\to\infty}$ in both sides of the above inequality, we obtain

$$\liminf_{n\to\infty}\left[\frac{\gamma(E_{i_0,n})}{n}\right] < \frac{1}{4\pi^2 p},$$

which implies by Theorem 1, that the function $f(z)$ is constant, and the corollary is proved.

2. A construction

In the introduction we mentioned that Riesz's theorem does not hold in full generality. In this section we are going to construct a function in AMP, nonconstant and such that it has the same nontangential limit in a set E of positive capacity.

Let $f(z)$ be the function constructed in [2]; it satisfies that $\iint_\Delta |f'(z)|^2\,dx\,dy < \infty$ and $\lim_{r\to 1} f(re^{i\theta}) = 0$ for all $e^{i\theta} \in E \subset \partial\Delta$ and $\operatorname{cap}(E) > 0$.

If we denote by $n(w)$ the number of roots of $f(z) = w$,

$$\iint_\Delta |f'(z)|^2\,dx\,dy = \int_0^\infty \int_0^{2\pi} n(w)\,d\sigma(w) < \infty,$$

where $d\sigma(w)$ denotes the Lebesgue measure. It follows from the absolute continuity of the above integral that for almost all complex w_0 we have

$$\lim_{r\to 1} \frac{1}{\pi r^2} \iint_{|w-w_0|<r} n(w)\,d\sigma(w) \to n(w_0) < \infty. \tag{2.1}$$

Choose w_0, w_1 so that (2.1) holds for both values, and set

$$F(z) = \frac{f(z) - w_0}{f(z) - w_1}.$$

Then $F(z)$ has angular limit $\alpha = w_0/w_1$ at all points of E. Also the equations $f(z) = 0, \infty$ only have finitely many roots, since $n(w_0)$ and $n(w_1)$ are finite. We claim that $f(z) \in \mathrm{AMP}$. In fact, it follows from (2.1) that if $N(w)$ denotes the number of roots of $F(z) = w$, then

$$\int_0^{2\pi} \int_0^R N(te^{i\phi})t\,dt\,d\phi = O(R^2) \tag{2.2}$$

as $R \to 0$ and

$$\int_0^{2\pi}\int_R^{\infty} \frac{N(te^{i\phi})t\,dt\,d\phi}{t^4} = O\left(\frac{1}{R^2}\right) \tag{2.3}$$

as $R \to \infty$. This implies

$$\int_{2^p}^{2^{p+1}}\int_0^{2\pi} N(te^{i\phi})t\,dt\,d\phi \le C4^p,$$

$-\infty < p < \infty$, where C is a positive constant. We use (2.2) for $p < 0$ and (2.3) for $p \ge 0$. Suppose now that $R > 0$, and choose q such that $2^q < R \le 2^{q+1}$. Then

$$\begin{aligned}\int_0^R\int_0^{2\pi} N(te^{i\phi})t\,dt\,d\phi &\le \sum_{p=-\infty}^{q}\int_{2^p}^{2^{p+1}}\int_0^{2^{p+1}} N(te^{i\phi})t\,dt\,d\phi \\ &\le C\sum_{p=-\infty}^{q} 4^p = \frac{4}{3}C4^q \le \frac{4}{3}CR^2.\end{aligned}$$

The function $F(z)$ can have a finite number of poles and zeros in Δ, but by using a conformal mapping of a cut annulus Ω onto the unit disc we can construct a function without poles and zeros. If we call this new function $F(z)$ again, then it is an areally mean $(\frac{4C}{3\pi})$-valent function, and $F(z)$ has the same angular limit $\alpha = w_0/w_1$ in a set E of positive capacity as we wanted to show.

Acknowledgment

The author expresses his gratitude to Professor Walter K. Hayman for his many invaluable comments concerning this paper and for providing him with the construction in §2.

References

1. A. Beurling, *Ensembles exceptionnels*, Acta Math. **72** (1940), 1–13.
2. L. Carleson, *Sets of uniqueness for functions analytic in the unit disc*, Acta Math. **87** (1952), 325–345.
3. E. F. Collingwood and A. J. Lohwater, *Theory of cluster sets*, University Press, London, 1966.
4. B. G. Eke, *The asymptotic behavior of areally mean p-valent functions*, J. Anal. Math. **20** (1967), 147–212.
5. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, London, 1967.
6. ——, Private communication.
7. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
8. Ch. Pommerenke, *On normal and automorphic functions*, Michigan Math. J. **21** (1974), 193–202.
9. M. Tsuji, *Beurling's theorem on exceptional sets*, Tôhoku Math. J. (2) **2** (1950), 113–125.
10. W. K. Hayman, *Subharmonic functions*, Vol. 2, Academic Press, London, 1990.

Department of Mathematics, Florida International University, Miami, Florida 33199
E-mail address: villamor@fiu.edu