PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 4, December 1994

HARDY INEQUALITIES AND IMBEDDINGS IN DOMAINS GENERALIZING $C^{0,\lambda}$ DOMAINS

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(Communicated by Andrew Bruckner)

ABSTRACT. Sufficient conditions for Hardy inequalities in a domain are studied together with related imbedding inequalities. The weights are nonnegative monotone functions of the distance to the boundary, and the domains are a generalization of $C^{0,\lambda}$ domains, obtained by replacing the power function in the definition with a general nondecreasing one.

INTRODUCTION

We treat Hardy inequalities and imbeddings for weighted Sobolev space in domains Ω in $\mathbf{R}^N$, which are generalizations of $C^{0,\lambda}$ domains. We use a general nondecreasing function instead of the power function in the corresponding definition of $C^{0,\lambda}$ domains. Furthermore also unbounded domains are considered. The weight functions v and w (below) are monotone nonnegative functions of $d_{\partial\Omega}(x)$, which denotes the distance to the boundary. The problem is to describe the possible Ω, v, and w. We give sufficient conditions. The constants A used in connection with (1), (2), and (3) below are independent of u.

First we describe one of the Hardy inequalities. (The other is a variation.) Let u be absolutely continuous along almost all lines, and let $u|_{\partial\Omega} = 0$. The inequality considered is

$$\|u\|_{L^p(\Omega, v)} \leq A\|\nabla u\|_{L^p(\Omega, w)}, \tag{1}$$

where $L^p(\Omega, w)$ is defined as the L^p-space on Ω with weight w, etc.

Next the Sobolev space $W^{1,p}(\Omega, w)$ is defined as the space of functions u on Ω such that the weak gradient is in $L^p(\Omega, w)$ and with the norm

$$\|u\|_{W^{1,p}(\Omega, w)} = \|\nabla u\|_{L^p(\Omega, w)} + \|u\|_{L^p(\Omega, w)}$$

finite. The inequality

$$\|u\|_{L^p(\Omega, v)} \leq A\|u\|_{W^{1,p}(\Omega, w)}$$

is equivalent to the fact that the identity map gives a continuous imbedding of $W^{1,p}(\Omega, w)$ into $L^p(\Omega, v)$. This is written

$$W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, v). \tag{2}$$

Received by the editors November 22, 1991 and, in revised form, April 2, 1993.
1991 *Mathematics Subject Classification.* Primary 26D15, 46E35, 35J30.

Let $W_0^{1,p}(\Omega, w)$ denote the closure of $C_0^\infty(\Omega)$, the set of functions with compact support in Ω and which are infinitely many times differentiable on Ω, in the norm $\| \ \|_{W^{1,p}(\Omega, w)}$.

If (1) holds then

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, v), \tag{3}$$

analogous to (2), follows trivially. But (3) can also be proved without making use of (1).

Our result is Theorem 8, and the proof makes use of a method, which is a development of the method used by Nečas in [12] and Kufner in [8]. The idea is to reduce the problem to a one-dimensional one and then apply what is known about the one-dimensional Hardy inequality. We use a theorem by Tomaselli in [15] that gives a necessary and sufficient condition on the weights for this inequality to hold. See also Muckenhaupt [11], Maz′ja [10, 1.3], Bradley [2], and Stredulinsky [14, 1.2.9].

Next we discuss the previous related results. The book [10], by Maz′ja contains numerous necessary and sufficient results that either have the inequalities (1) or (3) as special cases or as variations. We pick the following as being of special relevance: 2.1.1, 2.1.3, 2.3.2–2.3.3, and 2.3.7. A general problem with these theorems however, is, that it seems to be very difficult to check the respective necessary and sufficient conditions except for simple cases. Here 2.1.1 contains a simpler condition, which might be possible to check in more general cases. Stredulinsky has in [14, 2.2.41] given a result similar to that of Maz′ja [10, 2.3.3], with an easier proof, based on a different idea by Maz′ja.

Kadlec and Kufner proved in [7] that (1) holds for a bounded Lipschitz domain, with weights $v = d_{\partial\Omega}(x)^{-p+\delta}$ and $w = d_{\partial\Omega}(x)^\delta$, for $\delta < p - 1$. Actually the results are formulated for arbitrary order of the Sobolev space. The order one formulation is a special case of our main theorem.

In [8] Kufner has proved four theorems relevant to the discussion.

The theorems 8.2 and 8.4 hold for $C^{0,\lambda}$ domains. The weights considered are $d_{\partial\Omega}(x)$ to various powers. If we restrict the weights in our theorem to powers of $d_{\partial\Omega}(x)$ and study $C^{0,\lambda}$ domains, then the results agree, except that when A. Kufner gets a result of type (3) we can have a slightly stronger result of type (1) if the domain satisfies some mild condition. The theorems 12.7 and 12.9 concern bounded Lipschitz domains and weights that are monotone functions of $d_{\partial\Omega}(x)$. The results are less general than ours for Lipschitz domains, since they do not use a necessary and sufficient condition for the weights of the one-dimensional Hardy inequality.

The inequality (1) has been generalizerd by Opic and Kufner (see [13]) to a situation with different weight functions for the different partial derivatives of order one.

T. Horiuchi (see [6, Theorem 3]) has proved the imbedding (2) for weight functions, which are powers of $d_{\partial\Omega}(x)$, and for domains that have a Lipschitz manifold of arbitrary dimension as boundary. Thus the domains differ from those treated here, except for the Lipschitz domains. Furthermore, he allows for an imbedding into a weighted L^q-space with $q \geq p$.

After this paper was completed our attention was brought to three consecutive papers by Gurka and Opic [3-5], which give many results on imbeddings and compact imbeddings in weighted Sobolev space. The results of Kufner mentioned above (Theorems 8.2 and 8.4) are generalized to the corresponding cases with $q \neq p$ and $p \geq 1$. The theorem most relevant to the present paper is Theorem 6.1, which is a complicated sufficient condition for imbeddings of type (2). It seems to be involved to compare this with the corresponding situation in Theorem 8.

We treat (1) in [16], with weights $v = d_{\partial\Omega}(x)^{\delta-p}$ and $w = d_{\partial\Omega}(x)^{\delta}$, for $\delta < s_0$, where s_0 is some positive constant. Also higher-order gradients are considered. The condition on Ω for (1) to hold is that a certain capacity of the complement, defined locally at the boundary, should be bounded from below. An example is the planar case. It is then sufficient that Ω is simply connected. This shows that the main theorem of the present article does not give a necessary condition for (1).

The preliminary manuscript [17] contains [16] as a special case and also related material and questions.

For further references on the subject of the present paper and on applications to partial differential equations we refer to Kufner [8], Kufner and Sändig [9], Maz′ja [10], Horiuchi [6], and Avantaggiati [1].

Let A in the following denote a positive constant, which may change even within the same string of inequalities. Let $\sim$ denote that the quotients between LHS and RHS is bounded from above and below by positive constants. Furthermore $\gtrsim$ denotes that either the LHS $\sim$ RHS or LHS $>$ RHS, etc.

Results

We need some definitions in order to be able to state the main theorem. To begin, we define local coordinates in a general setting, i.e., also unbounded regions are allowed. The coordinate systems are orthogonal, and the cylinders that will be used are rotation symmetrical. (More general cylinders are possible.)

1. **Definition.** An open set Ω in $\mathbf{R}^N$ is said to *admit local coordinates* if the following set of conditions are satisfied:

(i) There is a countable set of open cylinders $\{C_i\}$ in $\mathbf{R}^N$. To every cylinder C_i there is associated a coordinate system $\{e_{in}\}_{n=1}^N$ such that the axis e_{iN} has the same direction as the axis for C_i.

(ii) There exist continuous functions $a_i(x_i')$, where $x = (x_i', x_{iN}) \in C_i$ and $x_i' = (x_{i1}, x_{i2}, \dots, x_{i(N-1)})$, and constants $\{M_i\}$ $(0 < M_i \leq \infty)$, with $\inf_i M_i > 0$. The following properties hold. Denote the set $\{x\colon x_{iN} = a_i(x_i'),\ x \in C_i\}$ by J_i, and denote the subset of C_i, formed by the union of all translates of J_i formed by $J_i + se_{iN}$, where for every s, $0 < s < M_i$, by F_i. Then $J_i = \partial\Omega \cap \overline{F}_i$ and $F_i \subset \Omega$.

(iii) There is a finite constant $T > 0$ such that if $x \in \Omega$ and $d_{\partial\Omega}(x) < T$ then $x \in F_i$ for some i.

For domains that admits local coordinates we denote $\sup_i M_i$ with M.

2. **Notation.** Suppose that an open set in $\mathbf{R}^N$, Ω, admits local coordinates. Then we denote the set $\bigcup_i F_i$ with $\Omega_{\mathscr{M}}$ and for a constant S write Ω_S for $\{x \in \Omega\colon d_{\partial\Omega}(x) < S\}$.

3. **Definition.** An open set Ω in $\mathbf{R}^N$ is said to *admit local coordinates of the uniform type* if Ω admits local coordinates such that:

(i) For $x \in F_i$ it is true that $d_{J_i}(x) \sim d_{\partial\Omega}(x)$, where $d_{J_i}(x)$ is the distance between x and J_i.

(ii) The sets $\{F_i\}$ cover $\Omega_{\mathscr{M}}$ locally finitely, i.e., there is a number $K > 0$, such that any intersection of $K+1$ of these sets is empty.

Next we give a generalization of $C^{0,\lambda}$ domains.

4. **Definition.** Let $f \geq 0$ be a nondecreasing function on the nonnegative half axis such that $\lim_{t\to 0} f(t) = 0$. An open set Ω in $\mathbf{R}^N$ is of class $C^{0,f}$ if it admits local coordinates of the uniform type and the $a_i(x_i')$ (see Definition 1) obey a Hölder-like condition as follows:

$$\sup_{x,y\in C_i} \frac{|a_i(x_i') - a_i(y_i')|}{f(|x_i' - y_i'|)} \leq 1.$$

Here C_i is as in Definition 1.

If $f(t) = C \cdot t^\lambda$, where C is a constant and $0 < \lambda \leq 1$, we say that Ω is of class $C^{0,\lambda}$.

5. **Definition.** Given a coordinate system $\{e_{in}\}_1^N$ and given a real function $k(x_i')$ defined for $x \in C_i$, where C_i is a cylinder with axis e_{iN}, denote the distance from the point $(x_i', k(x_i') + t \cdot e_{iN})$ to the graph of k with $\tilde{k}(x_i', t)$.

6. **Definition.** Given a function f as in Definition 4 and given a coordinate system $\{e_n\}_{n=1}^N$, define $\overline{f}(x') = f(|x'|)$.

7. **Definition.** Let ρ be a nonnegative monotone function on the positive halfaxis. Let f be as in Definition 4. Define $h_{f,\rho}(t) = \tilde{\overline{f}}(0, t)$ if ρ is nonincreasing and $h_{f,\rho}(t) = t$ if ρ is nondecreasing. Furthermore, define $H_{f,\rho}(t) = \tilde{\overline{f}}(0, t)$ if ρ is nondecreasing and $H_{f,\rho}(t) = t$ if ρ is nonincreasing. (There will be no ambiguity with ρ constant.)

Denote $p/(p-1)$ with p'.

8. **Theorem.** *Let Ω be an open set in $\mathbf{R}^N$ of class $C^{0,f}$. Let $1 \leq p$, and let ϕ and Φ be monotone functions on the positive half axis, with $\phi \geq 0$ and $\Phi > 0$. Assume that $A \cdot \phi(t) \geq \phi(\frac{1}{2}t)$ and $\Phi(t) \leq A \cdot \Phi(\frac{1}{2}t)$, for some constants A.*

First, let u be absolutely continuous on almost all lines in any direction, and let $u|_{\partial\Omega} = 0$ for almost all these lines. Assume that

$$\sup_{0<s<M} \left(\int_s^M \phi(h_{f,\phi})\, dt \right)^{1/p} \left(\int_0^s (\Phi(H_{f,\Phi}))^{-p'/p}\, dt \right)^{1/p'} < \infty \tag{4}$$

holds. Then

$$\int_{\Omega_{\mathscr{M}}} |u|^p \phi(d_{\partial\Omega}(x))\, dx \leq C \int_{\Omega_{\mathscr{M}}} |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx. \tag{i}$$

Second, let u be absolutely continuous on almost all lines in any direction, and let $u|_{\Omega^c} = 0$ for almost all these lines. Let Ω be bounded. Assume that $\inf_{x\in\Omega} \Phi(d_{\partial\Omega}(x)) > 0$, $\sup_{x\in\Omega\setminus\Omega_{\mathscr{M}}} \phi(d_{\partial\Omega}(x)) < \infty$, and (4) holds. Then

$$\int_{\Omega} |u|^p \phi(d_{\partial\Omega}(x))\, dx \leq C \int_{\Omega} |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx. \tag{ii}$$

(*The constants denoted C are independent of u.*)

Third, assume that there is a constant A such that $\phi(d_{\partial\Omega}(x)) \leq A \cdot \Phi(d_{\partial\Omega}(x))$ for $x \in \Omega \setminus \Omega_{\mathcal{M}}$ and that (4) *holds. Then*

$$\text{(iii)} \qquad W_0^{1,p}(\Omega, \Phi(d_{\partial\Omega}(x))) \hookrightarrow L^p(\Omega, \phi(d_{\partial\Omega}(x))).$$

Fourth, assume that there is a constant A such that $\phi(d_{\partial\Omega}(x)) \leq A \cdot \Phi(d_{\partial\Omega}(x))$ for $x \in \Omega \setminus \Omega_{\mathcal{M}}$ and that

$$(5) \qquad \sup_{0<s<M} \left(\int_0^s \phi(h_{f,\phi})\,dt \right)^{1/p} \left(\int_s^M (\Phi(H_{f,\Phi}))^{-p'/p}\,dt \right)^{1/p'} < \infty$$

holds. Then

$$\text{(iv)} \qquad W^{1,p}(\Omega, \Phi(d_{\partial\Omega}(x))) \hookrightarrow L^p(\Omega, \phi(d_{\partial\Omega}(x))).$$

Proof. (i) The set F_i is the disjoint union of line segments parallel to e_{iN}, each of length M_i. If L is such a line segment, we say that L is associated to F_i. Assume that we have proved the corresponding inequality for every L associated to F_i and for every i, i.e., we have that

$$(6) \qquad \int_L |u|^p \phi(d_{J_i}(x))\,dt \leq A \int_L |Du|^p \Phi(d_{J_i}(x))\,dt$$

for some constant A independent of L, u, and i. Here Du denotes the derivative of u along the line segment L.

By the Tonelli theorem and the finite intersection property of the local coordinates we have that

$$\int_{\Omega_{\mathcal{M}}} |u|^p \phi(d_{\partial\Omega}(x))\,dx \sim \sum_i \int_{\{x_i' \,:\, x \in C_i\}} \int_{a(x_i')}^{a(x_i')+M_i} |u|^p \phi(d_{\partial\Omega}(x))\,dx_{iN}\,dx_i'.$$

Since $A \cdot \phi(t) \geq \phi(\frac{1}{2}t)$, we have that $A \cdot \phi(t) \geq \phi(\frac{1}{2^n}t)$. By this and the conditions on Ω we have that the RHS is less than or equivalent to

$$\sum_i \int_{\{x_i' \,:\, x \in C_i\}} \int_{a(x_i')}^{a(x_i')+M_i} |u|^p \phi(d_{J_i}(x))\,dx_{iN}\,dx_i'.$$

Then by (6) we have that this expression is less than or equivalent to

$$\sum_i \int_{\{x_i' \,:\, x \in C_i\}} \int_{a(x_i')}^{a(x_i')+M_i} |Du|^p \Phi(d_{J_i}(x))\,dx_{iN}\,dx_i'.$$

In the same way by the conditions on Φ and Ω this expression is less than or equivalent to

$$\sum_i \int_{\{x_i' \,:\, x \in C_i\}} \int_{a(x_i')}^{a(x_i')+M_i} |Du|^p \Phi(d_{\partial\Omega}(x))\,dx_{iN}\,dx_i'.$$

We use the finite intersection property of the local coordinates and the Tonelli theorem to get that this expression is less than or equivalent to

$$\int_{\Omega_{\mathcal{M}}} |\nabla u|^p \Phi(d_{\partial\Omega}(x))\,dx.$$

Hence it suffices to prove the one-dimensional inequality above.

The proof of this depends on the one-dimensional Hardy inequality with general weights; see Tomaselli [15], Muckenhaupt [11], Maz′ja [10, 1.3], Bradley [2], and Stredulinsky [14, 1.2.9].

The results are that the inequality

$$\left(\int_0^\infty |u|^p v\,dt\right)^{1/p} \le C\left(\int_0^\infty |Du|^p w\,dt\right)^{1/p} \tag{7}$$

for u absolutely continuous, $v \ge 0$ and $w \ge 0$, holds for some constant, $C = C(w, v)$, in two cases.

First, if $u(0) = 0$, then (7) holds if and only if

$$B_1 = \sup_s \left(\int_s^\infty v\,dt\right)^{1/p}\left(\int_0^s w^{-p'/p}\,dt\right)^{1/p'} < \infty. \tag{8}$$

Second, if $\lim_{t\to\infty} u(t) = 0$, then (7) is true if and only if

$$B_2 = \sup_s \left(\int_0^s v\,dt\right)^{1/p}\left(\int_s^\infty w^{-p'/p}\,dt\right)^{1/p'} < \infty. \tag{9}$$

The constant in (7) can in the respective case be taken as $c \cdot B_i$, where c is independent of v and w.

These results are adapted to our situation if $v(t) = 0$, $w(t) = \infty$, and $u = \text{const}$ for $t > M$.

Now take an L associated to F_i. Let t be the distance from a point $x \in L$ to the boundary along L. Then, according to Definition 3, $d_{\partial\Omega}(x)$ for $x \in L$ is equivalent to $\tilde{a}_i(x_i', t)$.

From the argument above and the one-dimensional Hardy inequality it follows that it is enough to prove

$$\sup_{0<s<M}\left(\int_s^M \phi(\tilde{a}_i(x_i', t))\,dt\right)^{1/p}\left(\int_0^s (\Phi(\tilde{a}_i(x_i', t))^{-p'/p}\,dt\right)^{1/p'} < D$$

for all possible choices of $a_i(x_i')$, where D is independent of $a_i(x_i')$ and i. Now we claim that

$$\phi(h_{f,\phi}(t)) \ge \phi(\tilde{a}_i(x_i', t)) \tag{10}$$

and

$$\Phi(H_{f,\Phi}(t)) \le \Phi(\tilde{a}_i(x_i', t)). \tag{11}$$

These inequalities are proved by examining the four different cases.

In the first case ϕ is nonincreasing. Then $h_{f,\phi}(t) = \tilde{\tilde{f}}(0, t)$. We want to show that

$$\tilde{\tilde{f}}(0, t) \le \tilde{a}_i(x_i', t).$$

We fix x_i' and change the coordinate system $\{e_{in}\}_{n=1}^N$ by a unique translation to $\{g_{in}\}_{n=1}^N$ so that $(x_i', a(x_i'))$ is mapped onto $(0, 0)$ in the new coordinates. The function a_i is transformed to b_i. But the character of the Hölder-like condition defining $C^{0,f}$ domains does not change with this change of the coordinates. Hence we have that

$$b_i(y_i') = b_i(y_i') - b_i(0) \le f(|y_i' - 0|). \tag{12}$$

Now the function $\tilde{a}_i(x_i', t)$ transforms to $\tilde{b}_i(0, t)$, and they are equal by their definition. We have by (12) and the obvious geometry at hand that

$$\tilde{a}_i(x_i', t) = \tilde{b}_i(0, t) \geq \tilde{\tilde{f}}(0, t),$$

and since ϕ is nonincreasing, we get that

$$\phi(h_{f,\phi}(t)) = \phi(\tilde{\tilde{f}}(0, t)) \geq \phi(\tilde{a}_i(x_i', t)).$$

In the second case ϕ is nondecreasing. Then $h_{f,\phi}(t) = t$ and obviously $t \geq \tilde{a}_i(x_i', t)$, but then $\phi(t) \geq \phi(\tilde{a}_i(x_i', t))$, since ϕ is nondecreasing.

Now (11) follows by symmetry.

From (10), (11), and (4) we get

$$\begin{aligned} &\sup_{0<s<M} \left(\int_s^M \phi(\tilde{a}_i(x_i', t))\, dt \right)^{1/p} \left(\int_0^s (\Phi(\tilde{a}_i(x_i', t))\, dt)^{-p'/p} \right)^{1/p'} \\ &\quad \leq \sup_{0<s<M} \left(\int_s^M \phi(h_{f,\phi})\, dt \right)^{1/p} \left(\int_0^s (\Phi(H_{f,\Phi}))^{-p'/p}\, dt \right)^{1/p'} < \infty. \end{aligned}$$

This ends the proof of (i).

(ii) First we observe that if u is absolutely continuous along almost all lines and $\int |u|^p\, dx + \int |\nabla u|^p\, dx < \infty$, then $u \in W^{1,p}$; see Maz'ja [10, 1.1.3].

We denote $\max_{x \in \Omega \setminus \Omega_{\mathscr{M}}} \phi(d_{\partial\Omega}(x))$ with $\overline{\phi}$. First we use (i); then we use a trivial estimate, and finally we use a Poincaré inequality. The latter is justified by the facts that Ω is bounded and that $u|_{\Omega^c} = 0$. We have that

$$\begin{aligned} \int_\Omega |u|^p \phi(d_{\partial\Omega}(x))\, dx &\leq A \int_{\Omega_{\mathscr{M}}} |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx + \int_{\Omega \setminus \Omega_{\mathscr{M}}} |u|^p \phi(d_{\partial\Omega}(x))\, dx \\ &\leq A \int_\Omega |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx + \int_\Omega |u|^p \overline{\phi}\, dx \\ &\leq A \left(\int_\Omega |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx + \int_\Omega |\nabla u|^p \overline{\phi}\, dx \right). \end{aligned}$$

By the conditions on ϕ and Φ we have that the RHS is less than or equal to

$$A \int_\Omega |\nabla u|^p \Phi(d_{\partial\Omega}(x))\, dx.$$

This ends the proof of (ii).

(iii) It is enough to consider $u \in C_0^\infty(\Omega)$, which is dense in the space $W_0^{1,p}(\Omega, \Phi(d_{\partial\Omega}(x)))$. Let T be defined as in Definition 1. Then by this definition $\Omega_T \subset \Omega_{\mathscr{M}}$. Take a cut-off function $\Psi(t)$, with $\Psi \in C^\infty$, $\Psi(t) = 1$ for $t \leq T/2$, $\Psi(t) = 0$ for $t \geq T$, and $0 \leq \Psi(t) \leq 1$. (If $T = \infty$, we make a limit procedure, this is also done with respect to M_i if $\sup M_i = \infty$.) Since $\phi(d_{\partial\Omega}(x)) \leq A\Phi(d_{\partial\Omega}(x))$ in $\Omega \setminus \Omega_{\mathscr{M}}$, it follows that $\phi(d_{\partial\Omega}(x)) \leq A\Phi(d_{\partial\Omega}(x))$ in $(\Omega_{T/2})^c$ from the conditions $A \cdot \phi(t) \geq \phi(\frac{1}{2}t)$ and $\Phi(t) \leq A \cdot \Phi(\frac{1}{2}t)$.

Now by the triangle inequality, (i), the relationship above, and Leibnitz's rule

$$\left(\int_\Omega |u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$\le \left(\int_\Omega |\Psi(d_{\partial\Omega}(x))u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$+ \left(\int_\Omega |(1-\Psi(d_{\partial\Omega}(x)))u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$\le A\left(\int_{\Omega_{\mathscr{A}}} |\nabla(\Psi(d_{\partial\Omega}(x))u)|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$+ \left(\int_\Omega |(1-\Psi(d_{\partial\Omega}(x)))u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$\le A\left(\left(\int_\Omega |\nabla(\Psi(d_{\partial\Omega}(x))u)|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}\right.$$

$$\left. + \left(\int_\Omega |u|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}\right)$$

$$\le A\left(\left(\int_\Omega |\nabla u|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p} + \left(\int_\Omega |u|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}\right).$$

This ends the proof of (iii).

(iv) The functions in $W^{1,p}(\Omega)$ are (possibly after a redefinition on a set of Lebesgue measure zero) absolutely continuous on almost all line segments in Ω; see Maz′ja [10, 1.1.3].

It is enough to prove the statement for $W^{1,p}(\Omega_T\backslash\Omega_\varepsilon, \Phi(d_{\partial\Omega}(x)))$, where ε is a small positive number, which we later will let tend to zero. (If $T=\infty$, we make a limit procedure.) It follows that also in this space the functions are absolutely continuous along all lines, since $\Phi(d_{\partial\Omega}(x))$ is bounded from below by a positive constant in $\Omega_T\backslash\Omega_\varepsilon$. Then define a cut-off function $\Psi(t)$, just as in the previous case. We have by the triangle inequality that

$$\left(\int_{\Omega_T\backslash\Omega_\varepsilon} |u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$\le \left(\int_{\Omega_T\backslash\Omega_\varepsilon} |\Psi(d_{\partial\Omega}(x))u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}$$

$$+ \left(\int_{\Omega_T\backslash\Omega_\varepsilon} |(1-\Psi(d_{\partial\Omega}(x)))u|^p \phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}.$$

We observe that the second term in the RHS is less than or equal to

$$A\left(\int_{\Omega\backslash\Omega_\varepsilon} |u|^p \Phi(d_{\partial\Omega}(x))\,dx\right)^{1/p}.$$

Then we estimate the first term of the RHS. This is done by a procedure completely analogous to the case (i). Note that $\Psi(d_{\partial\Omega}(x))u$ is 0 outside Ω_T. See also case (iii). Then let ε tend to zero, and we have the desired result.

This ends the proof of (iv) and so the proof of Theorem 8.

We shall exemplify the results of Theorem 8 with the results for a bounded open set Ω in $\mathbf{R}^N$ of class $C^{0,\lambda}$, with $p > 1$ and with weights that are determined by $\phi(t) = t^\alpha$ and $\Phi(t) = t^\beta$. Furthermore, we assume that $\Omega_{\mathscr{M}} = \Omega$ and that $M \leq 1$. We observe that for domains of class $C^{0,\lambda}$, $\tilde{\bar{f}}(0, t) \sim t^{1/\lambda}$.

It is enough to examine case (i) and (iv). The calculations are only simple calculus and are omitted.

We have in case (i) that

$$\begin{aligned} &\beta \leq 0, && \alpha \geq \lambda(\beta - p), \\ &0 \leq \beta \leq \lambda(p-1), && \alpha \geq \beta - \lambda p. \end{aligned}$$

This agrees with the exponents arrived at by Kufner (see [8, Theorem 8.4]), for the type (3) case, i.e., we get Hardy inequalities where he gets imbeddings.

We have in case (iv) that

$$\begin{aligned} &\beta \leq \lambda(p-1), && \alpha > -\lambda, \\ &\lambda(p-1) < \beta \leq \lambda p, && \alpha \geq \beta - \lambda p, \\ &\lambda p \leq \beta, && \alpha \geq \beta/\lambda - p. \end{aligned}$$

This agrees with the exponents arrived at by Kufner (see [8, Theorem 8.2]) in the same case.

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