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WANDERING DOMAINS FOR INFINITELY RENORMALIZABLE DIFFEOMORPHISMS OF THE DISK

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ABSTRACT. Denjoy's theorem and counter-example for circle maps have counterparts for infinitely renormalizable diffeomorphisms of the 2-disk.

1. INTRODUCTION

A series of recent results in one- and two-dimensional dynamics (cf. [S, GST]) leads us to consider here some questions inspired by Denjoy's theory [B, D], aimed at furthering our understanding of the following form of dynamics (see Figure 1 on the next page). For the orientation preserving diffeomorphism f of the 2-disk:

(a) there is an invariant set K which breaks into a_1 disconnected closed pieces $K(i_1)$ cyclically permuted by f,
(b) each piece $K(i_1)$ of K breaks in a_2 disconnected closed pieces $K(i_1, i_2)$ cyclically permuted by $f_1 = f^{a_1}$,
(c) each piece $K(i_1, i_2)$ breaks in a_3 disconnected closed pieces $K(i_1, i_2, i_3)$ cyclically permuted by $f_2 = f_1^{a_2} = f^{a_2 \cdot a_1}$,
(d) continuing in this way, for all $n > 0$, $K(i_1, \dots, i_n)$ breaks similarly in a_{n+1} disconnected closed pieces $K(i_1, \dots, i_{n+1})$ cyclically permuted by $f_{n+1} = f_n^{a_{n+1}} = f^{a_n+1 \cdot \cdots \cdot a_1}$.

Such a map f is called *infinitely renormalizable.* The pair (K, f) is said to be *of type* $\{a_n\}_n$. Writing

$$\delta_{m,\alpha}(K, f) = \sum_{0 \le i_j < a_j} [\operatorname{diam}(K(i_1, \dots, i_m))]^\alpha$$

for $\alpha \geq 0$, we say that (K, f) has an *α-bounded geometry* if

$$\limsup_{m \to \infty} \delta_{m,\alpha}(K, f) < \infty.$$

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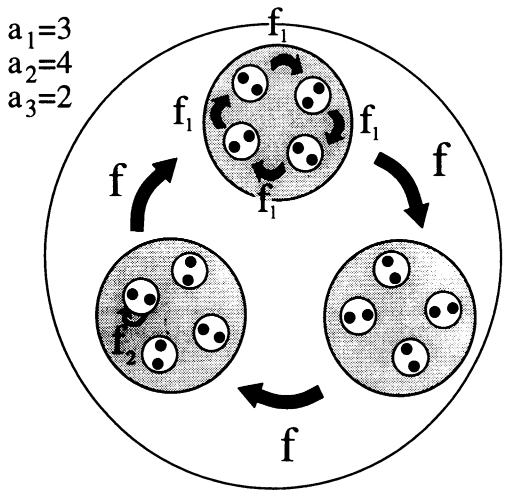

FIGURE 1

On the basis of Remark 3 below, we say that K contains a *wandering domain* if its interior is nonvoid.

Theorem 1. *Let f be a $C^{1+\alpha}$ infinitely renormalizable diffeomorphism of the 2-disk. If (K, f) has α-bounded geometry, with $0 \leq \alpha \leq 1$, then each connected component of K has zero Lebesgue measure. In particular, K does not contain any wandering domain.*

Theorem 2. *For any sequence $\{a_n\}_n$ with $a_n \geq 2$, there exists a C^1 infinitely renormalizable diffeomorphism f of the 2-disk such that (K, f) is of type $\{a_n\}_n$, with a 1-bounded geometry, and K contains a wandering domain.*

Remark 3. In the classical Denjoy theory, a wandering domain is a fibre with nonempty interior, of a semiconjugacy h from a homeomorphism f, of the circle $\mathbb{T}$, to a translation R_α of $\mathbb{T}$, with irrational angle α. This means that h: $\mathbb{T} \to \mathbb{T}$ is a continuous onto map such that $h \circ f = R_\alpha \circ h$ and, for some $x \in \mathbb{T}$, the set $h^{-1}(x)$ has nonempty interior, and we notice that the irrationality of α prevents the motion to be periodic. Our use of the words "wandering domain" involves two generalizations:

(1) the model map is a nonperiodic translation T on a compact abelian group G which is not necessarily the circle (see Remark 4),[1]
(2) if f is a homeomorphism of M, the semiconjugacy is not necessarily from f to T but from the restricted map $f_{|N}$ to T for some invariant subset $N \subset M$.

Remark 4. In our case, with the notation of Remark 3,

- G is $\widehat{\mathbb{Z}}_Q = \lim_{\leftarrow q_i} \mathbb{Z}/q_i\mathbb{Z}$ where Q stands for a super natural number $Q = \prod_p p^{k_p}$ where, $\forall p$ prime, $0 \leq k_p \leq \infty$, and the q_i's form a sequence of divisors of Q ordered by divisibility, and
- T is adding 1 on $G = \widehat{Z}_Q$, i.e., a generalized adding machine, where the usual adding machine corresponds to the case when $a_n \equiv 2$.

[1] This might be further generalized to other recurrent aperiodic motions.

2. Proof of Theorem 1

A connected component D_∞ of K is the intersection of an infinite sequence of nested $K(i_1, \cdots, i_m)$'s. Actually, we can always arrange the labels so that

$$D_\infty = \bigcap_{m\geq 0} K\underbrace{(0, \dots, 0)}_{m \text{ times}}.$$

Assume $\int_{D_\infty} dx \neq 0$, and let

$$u_n = \int_{D_\infty} \det D(f^n)(x)\, dx \quad \text{and} \quad v_n = \int_{D_\infty} \det D(f^{-n})(x)\, dx.$$

Since D_∞ never intersects its images under iteration, we have $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$. It follows that

$$\int_{D_\infty} \log(\det D(f^n)(x))\, dx \to -\infty \quad \text{when } n \to \pm\infty,$$

and thus

$$W_n = \sum_{i=0}^{n-1} \int_{D_\infty} [\log(\det D(f)(f^i(x))) - \log(\det D(f)(f^{i-n}(x))]\, dx \to -\infty \text{ as } n \to \infty.$$

Assume that $n = a_1 \cdot \cdots \cdot a_m$. Then for $0 \leq i \leq n-1$, $f^i(D_\infty)$ and $f^{i-n}(D_\infty)$ belong to the same $K(i_1, \dots, i_m)$, while $f^j(D_\infty)$ belongs to a different $K(i_1, \dots, i_m)$ for $j \notin \{i, i-n\}$ (see Figure 2).

If f is $C^{1+\alpha}$, then there exists some $C > 0$ such that, for all y and z in $\mathbb{D}^2$,

$$|\log(\det Df(y)) - \log(\det Df(z))| \leq C \cdot |y - z|^\alpha$$

so that $|W_n| \leq C \cdot \delta_{m,\alpha}(K, f) \cdot \int_{D_\infty} dx$. Thus, if (K, f) has α-bounded geometry, $|W_n|$ cannot go to ∞ as $n \to \infty$, a contradiction. □

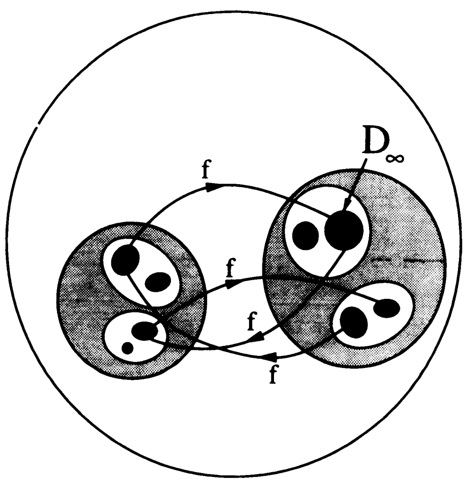

Figure 2

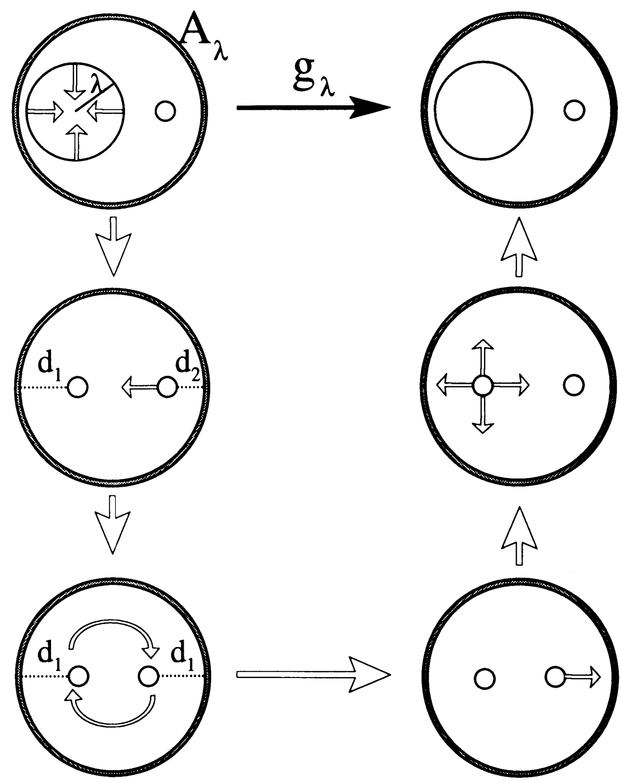

FIGURE 3

3. PROOF OF THEOREM 2

To simplify the formulas and the picture, we shall restrict here to the case when $a_n \equiv 2$. We are going to construct a map f as the limit in the C^1 topology (with norm $\|\cdot\|_1$) of a sequence of C^∞ orientation-preserving diffeomorphisms $\{f_n\}_n$ of the 2-disk $\mathbb{D}^2$. This construction is merely a modification of the one in [BF].

As the main building block of the construction, let us consider a smooth orientation-preserving diffeomorphism g_λ of $\mathbb{D}^2$, where $\frac{1}{2} \le \lambda < 1$, with the following properties (see Figure 3):

- g_λ is the identity in the annulus $\mathscr{A}_\lambda = \{x \in \mathbb{D}^2 | \, \|x\| > \frac{5+3\lambda}{8}\}$,
- $\mathbb{D}^2 \backslash \mathscr{A}_\lambda$ contains two disjoint disks, D_λ with radius λ and D'_λ with radius $\frac{1-\lambda}{4}$, such that g_λ maps each of these inner disks to the other as a composition of homotheties, translations, and rotations.

We regroup the following statements in the form of a lemma, leaving the details of the proof to the reader (see Figure 3):

Lemma. *We can choose the family of maps* $\{g_\lambda\}_{\lambda \in [1/2,1)}$, *and for each* λ *an isotopy* g_λ^t *from the identity* $\mathrm{Id}_{\mathbb{D}^2}$ *to* g_λ *so that*:

(i) g_λ^t *is the identity in* $\mathscr{A}_\lambda$,

(ii) g_λ^t *restricted to both* D_λ *and* D'_λ, *is a composition of homotheties, translations, and rotations,*

(iii) *there exists a constant* $K > 0$ *such that* $\|g_\lambda - \mathrm{Id}_{\mathbb{D}^2}\|_1 \le K/(1-\lambda)$ *and*

$$\|g_\lambda^{(i+1)/N} \circ (g_\lambda^{i/N})^{-1} - \mathrm{Id}_{\mathbb{D}^2}\|_1 \le \frac{K}{N\cdot(1-\lambda)} \quad \textit{for all } \lambda \textit{ in } [\tfrac{1}{2}, 1),\ N > 0, \textit{ and } 0 \le i < N.$$

Now choose a sequence $(\lambda_m)_{m>0}$, with $\lambda_m \to 1$, and a sequence of affine scalings $(\Lambda_m)_{m>0}$ such that Λ_m maps D_{λ_m} onto $\mathbb{D}^2$. We also define the affine scalings A_m and the disks D_m by

$$A_m = \Lambda_1 \circ \cdots \circ \Lambda_m \quad \text{and} \quad D_m = A_m^{-1}(\mathbb{D}^2).$$

Everything is now in place for the construction. We set $f_1 = g_{\lambda_1}$ and define inductively the sequence $\{f_n\}_n$ by setting

$$f_{m+1} = \begin{cases} f_m^{i+1} \circ A_m^{-1} \circ g_{\lambda_{m+1}}^{(i+1)/2^m} \circ (g_{\lambda_{m+1}}^{i/2^m})^{-1} \circ A_m \circ f_m^{-i} & \text{for } x \in f_m^i(D_m), \\ & i = 0, \dots, 2^m - 1, \\ f_m \quad \text{elsewhere.} \end{cases}$$

The map f_m is a C^∞ diffeomorphism that cyclically permutes 2^m disks $f_m^i(D_m)$ for $i = 0, \dots, 2^m - 1$ and the restriction of $f_m^{2^m}$ to D_m is the identity.

The definition of f_{m+1} is such that it differs from f_m only on the disks $f_m^i(D_m)$, where

$$(*) \qquad f_{m+1} - f_m = f_m^{i+1} \circ A_m^{-1} \circ (g_{\lambda_{m+1}}^{(i+1)/2^m} \circ (g_{\lambda_{m+1}}^{i/2^m})^{-1} - \mathrm{Id}_{|\mathbb{D}^2}) \circ A_m \circ f_m^{-i},$$

and since the restriction of f_m to the disks $f_m^i(D_m)$ is a composition of homotheties, translations, and rotations, it follows from $(*)$ and the lemma that

$$(**) \qquad \|f_{m+1} - f_m\|_1 \le \frac{K}{2^m \cdot (1-\lambda_{m+1})} \cdot \|f_m\|_1.$$

We now choose a sequence λ_m which goes fast enough to 1 to get a wandering domain but not too fast to keep control of the C^1 norm: for instance $\lambda_m = 1 - 1/(m+1)^2$.

We get

$$\|f_{m+1} - f_m\|_1 \le K \cdot \frac{(m+2)^2}{2^m} \cdot \|f_m\|_1$$

from $(**)$; thus,

$$\|f_{m+1}\|_1 \le \prod_{i=1}^{m} \left(1 + \frac{K\cdot(i+2)^2}{2^i}\right) \cdot \|f_1\|_1.$$

The product $\prod_{i=1}^m (1 + (K\cdot(i+1)^2)/2^i)$ converges to a finite limit as $m \to \infty$, as easily follows from $\prod_{k=0}^\infty (1 + x^{2^k}) = 1/(1-x)$ for $|x| < 1$. Consequently $\|f_m\|_1$ is uniformly bounded by a constant $C > 0$. Using $(**)$ again, we have

$$\|f_{m+1} - f_m\|_1 \le C \cdot K \cdot (m+2)^2/2^m.$$

It follows that $\{f_n\}_n$ is a Cauchy sequence in the C^1 topology and, hence, converges to a C^1 limit map. Throughout the construction, one can control that the inverse maps f_n^{-1} converge as well in the C^1 topology so that f is a C^1 diffeomorphism of $\mathbb{D}^2$.

Let

$$K = \bigcap_{m \geq 0} \bigcup_{i=0}^{2^m - 1} f^i(D_m).$$

Then f is a C^1 infinitely renormalizable diffeomorphism and (K, f) is of type $\{a_n\}_n$ with $a_n \equiv 2$. Since $\bigcap_{m \geq 0} D_m$ is a disk with radius $\prod_{m=1}^{\infty} \lambda_m = \prod_{k=2}^{\infty}(1 - 1/k^2) = \frac{1}{2} > 0$, f has a wandering domain. Finally, since in this construction, we have

$$\forall m \geq 1, \qquad \delta_{m+1,1} < \delta_{m,1};$$

is is obvious that (K, f) has a 1-bounded geometry. □

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References

[B] P. Bohl, *Uber die hinsichtlich der unabhängigen variabeln periodische Differentialgleichung erster Ordnung*, Acta Math. **40** (1916), 321–336.

[BF] R. Bowen and J. Franks, *The periodic points of maps of the disk and the interval*, Topology **15** (1976), 337–342.

[D] A. Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore*, J. Math. Pures Appl. (9) **11** (1932), 333–375.

[GST] J. M. Gambaudo, S. van Strien, and C. Tresser, *The periodic orbit structure of orientation preserving diffeomorphisms on* $\mathbb{D}^2$ *with topological entropy zero*, Ann. Inst. H. Poincaré Phys. Théor. **49** (1989), 335–356.

[S] D. Sullivan, *Bounds, quadratic differentials, and renormalization conjectures,* Mathematics into the Twenty-first Century, Vol. II, Amer. Math. Soc. Centennial Publication, Amer. Math. Soc., Providence, RI, 1992, pp. 417–466.

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