FIXED POINT THEOREM
FOR NONEXPANSIVE SEMIGROUPS ON BANACH SPACE

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(Communicated by Palle E. T. Jorgensen)

Abstract. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space, and let $S$ be a semitopological semigroup such that $\text{RUC}(S)$ has a left invariant submean. Then we prove a fixed point theorem for a continuous representation of $S$ as nonexpansive mappings on $C$.

1. Introduction

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to sa$ and $s \to as$ from $S$ into $S$ are continuous. Let $C$ be a nonempty subset of a Banach space $E$, and let $\mathcal{F} = \{T_t : t \in S\}$ be a family of self-maps of $C$. Then $\mathcal{F}$ is said to be a continuous representation of $S$ as nonexpansive mappings on $C$ if the following conditions are satisfied:

1. $T_{st}x = T_s T_t x$ for all $t, s \in S$ and $x \in C$;
2. the mapping $(s, x) \to T_s x$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology.

Fixed point theorems for a continuous representation of $S$ as nonexpansive mappings on $C$ have been investigated by several authors; see, for example, Bartoszek [2], Lau [3], Lau and Takahashi [4, 5], Mizoguchi and Takahashi [7], Takahashi [8–10], Tan and Xu [11], Xu [12], and others. Recently, Lau and Takahashi [5] and Tan and Xu [11] proved fixed point theorems for such a representation in the case of which $E$ is a uniformly convex Banach space and $\text{RUC}(S)$, the space of bounded right uniformly continuous functions on $S$, has a left invariant mean. On the other hand, Mizoguchi and Takahashi [7] introduced the notion of submean which generalizes "mean" and "limsup" and proved a fixed point theorem in a Hilbert space which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups.

Received by the editors April 1, 1993.
1991 Mathematics Subject Classification. Primary 47H10.
Key words and phrases. Fixed point, nonexpansive mapping, mean.
This paper was prepared during the second author's stay at Tokyo Institute of Technology under financial support by Korea Science and Engineering Foundation, 1991.

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In this paper, we prove a fixed point theorem for a continuous representation \( \mathcal{F} \) in the case of which \( E \) is a uniformly convex Banach space and \( \text{RUC}(\mathcal{S}) \) has a left invariant submean. This theorem generalizes the results [9, 5, 11].

2. Fixed point theorem

Let \( S \) be a set, and let \( m(S) \) be the Banach space of all bounded real-valued functions on \( S \) with supremum norm. Let \( X \) be a subspace of \( m(S) \) containing constants. A real-valued function \( \mu \) on \( X \) is called a submean on \( X \) [7] if the following conditions are satisfied:

1. \( \mu(f + g) \leq \mu(f) + \mu(g) \) for every \( f, g \in X \);
2. \( \mu(\alpha f) = \alpha \mu(f) \) for every \( f \in X \) and \( \alpha \geq 0 \);
3. for \( f, g \in X \), \( f \leq g \) implies \( \mu(f) \leq \mu(g) \);
4. \( \mu(c) = c \) for every constant \( c \).

Let \( \mu \) be a submean on \( X \) and \( f \in X \). Then, according to time and circumstances, we use \( \mu_t(f(t)) \) instead of \( \mu(f) \). The following proposition is used to prove the lemma.

Proposition (cf. [12, 13]). Let \( p > 1 \) and \( b > 0 \) be two fixed numbers. Then a Banach space \( E \) is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function (depending on \( p \) and \( b \)) \( g : [0, \infty) \rightarrow [0, \infty) \) such that \( g(0) = 0 \) and

\[
\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)
\]

for all \( x, y \in B_b \) and \( 0 \leq \lambda \leq 1 \), where \( W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda) \) and \( B_b \) is the closed ball with radius \( b \) and centered at the origin.

Lemma. Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space, let \( S \) be an index set, and let \( \{x_t : t \in S\} \) be a bounded set of \( E \). Let \( X \) be a subspace of \( m(S) \) containing constants, and let \( \mu \) be a submean on \( X \). Suppose that for each \( x \in C \) the real-valued function on \( S \) defined by

\[
f(t) = \|x_t - x\|^2 \quad \text{for all } t \in S
\]

belongs to \( X \). If \( r(x) = \mu_t\|x_t - x\|^2 \) for all \( x \in C \) and \( r = \inf\{r(x) : x \in C\} \), then there exists a unique element \( z \in C \) such that \( r(z) = r \).

Proof. We first prove that the real-valued function \( r \) on \( C \) is continuous and convex. Let \( x_n \rightarrow x \) and

\[
M = \sup\{\|x_t - x_n\| + \|x_t - x\| : n = 1, 2, \ldots, t \in S\}.
\]

Then, since

\[
\|x_t - x_n\|^2 - \|x_t - x\|^2 = (\|x_t - x_n\| + \|x_t - x\|)(\|x_t - x_n\| - \|x_t - x\|)
\leq M\|x_t - x_n\| - \|x_t - x\| \leq M\|x_n - x\|
\]

for every \( n = 1, 2, \ldots \) and \( t \in S \), we have

\[
\mu_t\|x_t - x_n\|^2 \leq \mu_t\|x_t - x\|^2 + M\|x_n - x\|.
\]

Similarly we have

\[
\mu_t\|x_t - x\|^2 \leq \mu_t\|x_t - x_n\|^2 + M\|x_n - x\|.
\]
So we have $|r(x_n) - r(x)| \leq M\|x_n - x\|$. This implies that $r$ is continuous on $C$. Let $\alpha$ and $\beta$ be nonnegative numbers with $\alpha + \beta = 1$ and $x, y \in C$. Then, since
\[
\|x_t - (\alpha x + \beta y)\|^2 \leq \alpha\|x_t - x\|^2 + \beta\|x_t - y\|^2,
\]
we have $r(\alpha x + \beta y) \leq \alpha r(x) + \beta r(y)$. This implies that $r$ is convex. We can also prove that if $\|x_n\| \to \infty$, then $r(x_n) \to \infty$. In fact, since
\[
\|x_n\|^2 \leq 2\|x_n - x_1\|^2 + 2\|x_1\|^2,
\]
we have
\[
\|x_n\|^2 \leq 2r(x_n) + 2M',
\]
where $M' = \sup\{\|x_t\|^2 : t \in S\}$. So if $\|x_n\| \to \infty$, then $r(x_n) \to \infty$. Therefore, it follows from [1, p. 79] that there is an element $z \in C$ with $r(z) = r$. Next we show that such an element $z \in C$ is unique. Let $K = \{z \in C : r(z) = r\}$. Then it is obvious that $K$ is nonempty, closed, and convex. Further, $K$ is bounded. In fact, let $z \in K$. Then since
\[
\|z\|^2 < 2\|z - x_1\|^2 + 2\|x_1\|^2,
\]
we have
\[
\|z\|^2 \leq 2r(z) + 2M' = 2r + 2M'.
\]
Choose $a > 0$ large enough so that $\{x_t : t \in S\} \cup K \subset B_a$, and put $b = 2a$. Then since $x_t - z_1, x_t - z_2 \in B_b$ for all $t \in S$ and $z_1, z_2 \in K$, from the proposition, we have
\[
\|x_t - \frac{1}{2}(z_1 + z_2)\|^2 \leq \frac{1}{2}\|x_t - z_1\|^2 + \frac{1}{2}\|x_t - z_2\|^2 - \frac{1}{4}g(\|z_1 - z_2\|).
\]
So if $z_1 \neq z_2$, we have
\[
r(\frac{1}{2}(z_1 + z_2)) \leq \frac{1}{2}r(z_1) + \frac{1}{2}r(z_2) - \frac{1}{4}g(\|z_1 - z_2\|) < r.
\]
This is a contradiction. Therefore, there exists a unique element $z \in C$ with $r(z) = r$.

Let $S$ be a semitopological semigroup. For $s \in S$ and $f \in m(S)$, we define $(lsf)(t) = f(st)$ and $(rsf)(t) = f(ts)$ for all $t \in S$. Let $X$ be a subspace of $m(S)$ containing constants which is $l_s$-invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. Then a submean $\mu$ on $X$ is said to be left invariant if $\mu(f) = \mu(l_sf)$ for all $s \in S$ and $f \in X$. Let $C(S)$ be the Banach space of bounded continuous real-valued functions on $S$. Let $RUC(S)$ denote the space of bounded right uniformly continuous functions on $S$, i.e., all $f \in C(S)$ such that the mapping $s \to rsf$ of $S$ into $C(S)$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under left and right translations (see [6] for details). A semitopological semigroup $S$ is left reversible if any two closed right ideals of $S$ have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on $S$ is defined by $a \leq b$ if and only if $\{a\} \cup aS \supset \{b\} \cup bS$. Now we can prove a fixed point theorem for a continuous representation of $S$ such that $RUC(S)$ has a left invariant submean.

**Theorem.** Let $S$ be a semitopological semigroup. Let $\mathcal{F} = \{T_t : t \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of a uniformly convex Banach space $E$ into $C$. If $RUC(S)$ has a left
invariant submean \( \mu \) and \( C \) contains an element \( u \) such that \( \{T_t u : t \in S\} \) is bounded, then there exists \( z \in C \) such that \( T_s z = z \) for all \( s \in S \).

**Proof.** First observe that for each \( y \in C \) the function \( h \) defined by

\[
h(t) = \|T_t u - y\|^2 \text{ for all } t \in S
\]

belongs to \( \text{RUC}(S) \). In fact, as Lau and Takahashi [5], we have, for all \( a, b \in S \),

\[
\|r_a h - r_b h\| = \sup_{t \in S} |(r_a h)(t) - (r_b h)(t)| = \sup_{t \in S} |h(ta) - h(tb)|
\]

\[
= \sup_{t \in S} \|T_{ta} u - y\|^2 - \|T_{tb} u - y\|^2\|
\]

\[
\leq k \sup_{t \in S} \|T_{ta} u - T_{tb} u\| \leq k \|T_a u - T_b u\|
\]

where \( k = 2 \sup_{t \in S}(\|T_t u\| + \|y\|) \). Then, \( h \in \text{RUC}(S) \).

Let \( \mu \) be a left invariant submean on \( \text{RUC}(S) \). Then the set

\[
K = \{z \in C : \mu(T_t u - z)^2 = \min_{y \in C} \mu(T_t u - y)^2\}
\]

is invariant under every \( T_s, s \in S \). In fact, if \( z \in K \), then for each \( s \in S \) we have

\[
\mu(T_s u - T_s z)^2 = \mu(T_{st} u - T_s z)^2 = \mu(T_s T_t u - T_s z)^2 \leq \mu(T_t u - z)^2
\]

and hence \( T_s z \in K \). On the other hand, by Lemma, we know that \( K \) consists of one point. Therefore, this point is a common fixed point of \( T_s, s \in S \).

The following two results, proved by the different methods, are deduced as the corollaries of Theorem.

**Corollary 1** (cf. [4, 12]). Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \), and let \( \mathcal{S} = \{T_t : t \in S\} \) be a continuous representation of \( S \) as nonexpansive mappings on \( C \). Suppose that \( \{T_t u : t \in S\} \) is bounded for some \( u \in C \) and \( \text{RUC}(S) \) has a left invariant mean. Then there exists \( z \in C \) such that \( T_s z = z \) for all \( s \in S \).

**Proof.** A left invariant mean \( \mu \) on \( \text{RUC}(S) \) is a left invariant submean on \( \text{RUC}(S) \). Therefore, from Theorem, the proof is complete.

**Corollary 2** (cf. [9]). Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \), and let \( \mathcal{S} = \{T_t : t \in S\} \) be a continuous representation of \( S \) as nonexpansive mappings on \( C \). Suppose that \( \{T_t u : t \in S\} \) is bounded for some \( u \in C \) and \( S \) is left reversible. Then there exists \( z \in C \) such that \( T_s z = z \) for all \( s \in S \).

**Proof.** Defining a real-valued function \( \mu \) on \( \text{RUC}(S) \) by

\[
\mu(f) = \limsup_{s} f(s) \text{ for every } f \in \text{RUC}(S),
\]

\( \mu \) is a left invariant submean on \( \text{RUC}(S) \). By using Theorem, the proof is complete.
References


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