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FIXED POINT THEOREM FOR NONEXPANSIVE SEMIGROUPS ON BANACH SPACE

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ABSTRACT. Let C be a nonempty closed convex subset of a uniformly convex Banach space, and let S be a semitopological semigroup such that $\mathrm{RUC}(S)$ has a left invariant submean. Then we prove a fixed point theorem for a continuous representation of S as nonexpansive mappings on C.

1. INTRODUCTION

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to sa$ and $s \to as$ from S into S are continuous. Let C be a nonempty subset of a Banach space E, and let $\mathscr{F} = \{T_t : t \in S\}$ be a family of self-maps of C. Then $\mathscr{F}$ is said to be a continuous representation of S as nonexpansive mappings on C if the following conditions are satisfied:

(1) $T_{st}x = T_sT_tx$ for all $t, s \in S$ and $x \in C$;
(2) the mapping $(s, x) \to T_sx$ from $S \times C$ into C is continuous when $S \times C$ has the product topology.

Fixed point theorems for a continuous representation of S as nonexpansive mappings on C have been investigated by several authors; see, for example, Bartoszek [2], Lau [3], Lau and Takahashi [4, 5], Mizoguchi and Takahashi [7], Takahashi [8–10], Tan and Xu [11], Xu [12], and others. Recently, Lau and Takahashi [5] and Tan and Xu [11] proved fixed point theorems for such a representation in the case of which E is a uniformly convex Banach space and $\mathrm{RUC}(S)$, the space of bounded right uniformly continuous functions on S, has a left invariant mean. On the other hand, Mizoguchi and Takahashi [7] introduced the notion of submean which generalizes "mean" and "limsup" and proved a fixed point theorem in a Hilbert space which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups.

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In this paper, we prove a fixed point theorem for a continuous representation $\mathscr{F}$ in the case of which E is a uniformly convex Banach space and RUC(S) has a left invariant submean. This theorem generalizes the results [9, 5, 11].

2. Fixed point theorem

Let S be a set, and let $m(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. Let X be a subspace of $m(S)$ containing constants. A real-valued function μ on X is called a *submean on* X [7] if the following conditions are satisfied:

(1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
(3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
(4) $\mu(c) = c$ for every constant c.

Let μ be a submean on X and $f \in X$. Then, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. The following proposition is used to prove the lemma.

Proposition (cf. [12, 13]). *Let $p > 1$ and $b > 0$ be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function (depending on p and b) $g: [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_b$ and $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and B_b is the closed ball with radius b and centered at the origin.

Lemma. *Let C be a nonempty closed convex subset of a uniformly convex Banach space, let S be an index set, and let $\{x_t : t \in S\}$ be a bounded set of E. Let X be a subspace of $m(S)$ containing constants, and let μ be a submean on X. Suppose that for each $x \in C$ the real-valued function on S defined by*

$$f(t) = \|x_t - x\|^2 \quad \textit{for all } t \in S$$

belongs to X. If

$$r(x) = \mu_t\|x_t - x\|^2 \quad \textit{for all } x \in C$$

and $r = \inf\{r(x) : x \in C\}$, then there exists a unique element $z \in C$ such that $r(z) = r$.

Proof. We first prove that the real-valued function r on C is continuous and convex. Let $x_n \to x$ and

$$M = \sup\{\|x_t - x_n\| + \|x_t - x\| : n = 1, 2, \ldots, t \in S\}.$$

Then, since

$$\begin{aligned}\|x_t - x_n\|^2 - \|x_t - x\|^2 &= (\|x_t - x_n\| + \|x_t - x\|)(\|x_t - x_n\| - \|x_t - x\|) \\ &\leq M|\,\|x_t - x_n\| - \|x_t - x\|\,| \leq M\|x_n - x\|\end{aligned}$$

for every $n = 1, 2, \ldots$ and $t \in S$, we have

$$\mu_t\|x_t - x_n\|^2 \leq \mu_t\|x_t - x\|^2 + M\|x_n - x\|.$$

Similarly we have

$$\mu_t\|x_t - x\|^2 \leq \mu_t\|x_t - x_n\|^2 + M\|x_n - x\|.$$

So we have $|r(x_n) - r(x)| \leq M\|x_n - x\|$. This implies that r is continuous on C. Let α and β be nonnegative numbers with $\alpha + \beta = 1$ and $x, y \in C$. Then, since

$$\|x_t - (\alpha x + \beta y)\|^2 \leq \alpha\|x_t - x\|^2 + \beta\|x_t - y\|^2,$$

we have $r(\alpha x + \beta y) \leq \alpha r(x) + \beta r(y)$. This implies that r is convex. We can also prove that if $\|x_n\| \to \infty$, then $r(x_n) \to \infty$. In fact, since

$$\|x_n\|^2 \leq 2\|x_n - x_t\|^2 + 2\|x_t\|^2,$$

we have

$$\|x_n\|^2 \leq 2r(x_n) + 2M',$$

where $M' = \sup\{\|x_t\|^2 : t \in S\}$. So if $\|x_n\| \to \infty$, then $r(x_n) \to \infty$. Therefore, it follows from [1, p. 79] that there is an element $z \in C$ with $r(z) = r$. Next we show that such an element $z \in C$ is unique. Let $K = \{z \in C : r(z) = r\}$. Then it is obvious that K is nonempty, closed, and convex. Further, K is bounded. In fact, let $z \in K$. Then since

$$\|z\|^2 \leq 2\|z - x_t\|^2 + 2\|x_t\|^2 \quad \text{for all } t \in S,$$

we have

$$\|z\|^2 \leq 2r(z) + 2M' = 2r + 2M'.$$

Choose $a > 0$ large enough so that $\{x_t : t \in S\} \cup K \subset B_a$, and put $b = 2a$. Then since $x_t - z_1, x_t - z_2 \in B_b$ for all $t \in S$ and $z_1, z_2 \in K$, from the proposition, we have

$$\|x_t - \tfrac{1}{2}(z_1 + z_2)\|^2 \leq \tfrac{1}{2}\|x_t - z_1\|^2 + \tfrac{1}{2}\|x_t - z_2\|^2 - \tfrac{1}{4}g(\|z_1 - z_2\|).$$

So if $z_1 \neq z_2$, we have

$$r(\tfrac{1}{2}(z_1 + z_2)) \leq \tfrac{1}{2}r(z_1) + \tfrac{1}{2}r(z_2) - \tfrac{1}{4}g(\|z_1 - z_2\|) < r.$$

This is a contradiction. Therefore, there exists a unique element $z \in C$ with $r(z) = r$.

Let S be a semitopological semigroup. For $s \in S$ and $f \in m(S)$, we define $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $m(S)$ containing constants which is l_s-invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. Then a submean μ on X is said to be left invariant if $\mu(f) = \mu(l_s f)$ for all $s \in S$ and $f \in X$. Let $C(S)$ be the Banach space of bounded continuous real-valued functions on S. Let RUC(S) denote the space of bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \to r_s f$ of S into $C(S)$ is continuous. Then RUC(S) is a closed subalgebra of $C(S)$ containing constants and invariant under left and right translations (see [6] for details). A semitopological semigroup S is left reversible if any two closed right ideals of S have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{aS} \supset \{b\} \cup \overline{bS}$. Now we can prove a fixed point theorem for a continuous representation of S such that RUC(S) has a left invariant submean.

Theorem. *Let S be a semitopological semigroup. Let $\mathscr{F} = \{T_t : t \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of a uniformly convex Banach space E into C. If* RUC(S) *has a left*

invariant submean μ *and* C *contains an element* u *such that* $\{T_t u : t \in S\}$ *is bounded, then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in S$.

Proof. First observe that for each $y \in C$ the function h defined by

$$h(t) = \|T_t u - y\|^2 \quad \text{for all } t \in S$$

belongs to $\mathrm{RUC}(S)$. In fact, as Lau and Takahashi [5], we have, for all $a, b \in S$,

$$\begin{aligned}\|r_a h - r_b h\| &= \sup_{t \in S} |(r_a h)(t) - (r_b h)(t)| = \sup_{t \in S} |h(ta) - h(tb)| \\ &= \sup |\, \|T_{ta} u - y\|^2 - \|T_{tb} u - y\|^2 | \\ &\le k \sup_{t \in S} \|T_{ta} u - T_{tb} u\| \le k \|T_a u - T_b u\|,\end{aligned}$$

where $k = 2 \sup_{t \in S} (\|T_t u\| + \|y\|)$. Then, $h \in \mathrm{RUC}(S)$.

Let μ be a left invariant submean on $\mathrm{RUC}(S)$. Then the set

$$K = \{z \in C : \mu_t \|T_t u - z\|^2 = \min_{y \in C} \mu_t \|T_t u - y\|^2\}$$

is invariant under every T_s, $s \in S$. In fact, if $z \in K$, then for each $s \in S$ we have

$$\mu_t \|T_t u - T_s z\|^2 = \mu_t \|T_{st} u - T_s z\|^2 = \mu_t \|T_s T_t u - T_s z\|^2 \le \mu_t \|T_t u - z\|^2$$

and hence $T_s z \in K$. On the other hand, by Lemma, we know that K consists of one point. Therefore, this point is a common fixed point of T_s, $s \in S$.

The following two results, proved by the different methods, are deduced as the corollaries of Theorem.

Corollary 1 (cf. [4, 12]). *Let* C *be a closed convex subset of a uniformly convex Banach space* E, *and let* $\mathscr{F} = \{T_t : t \in S\}$ *be a continuous representation of* S *as nonexpansive mappings on* C. *Suppose that* $\{T_t u : t \in S\}$ *is bounded for some* $u \in C$ *and* $\mathrm{RUC}(S)$ *has a left invariant mean. Then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in S$.

Proof. A left invariant mean μ on $\mathrm{RUC}(S)$ is a left invariant submean on $\mathrm{RUC}(S)$. Therefore, from Theorem, the proof is complete.

Corollary 2 (cf. [9]). *Let* C *be a closed convex subset of a uniformly convex Banach space* E, *and let* $\mathscr{F} = \{T_t : t \in S\}$ *be a continuous representation of* S *as nonexpansive mappings on* C. *Suppose that* $\{T_t u : t \in S\}$ *is bounded for some* $u \in C$ *and* S *is left reversible. Then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in S$.

Proof. Defining a real-valued function μ on $\mathrm{RUC}(S)$ by

$$\mu(f) = \limsup_s f(s) \quad \text{for every } f \in \mathrm{RUC}(S),$$

μ is a left invariant submean on $\mathrm{RUC}(S)$. By using Theorem, the proof is complete.

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