SOME ESTIMATES OF THE KOBAYASHI METRIC IN THE NORMAL DIRECTION

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Abstract. In this paper, we study the behavior of the Kobayashi metric in the normal direction near a Levi-pseudoconvex boundary point of a smoothly bounded domain without assuming global pseudoconvexity. As a corollary, we obtain a characterization of pseudoconvexity by the rate of the growth of the Kobayashi metric in the normal direction.

I. Introduction and theorems

Let \( \Omega \subset \subset \mathbb{C}^n \) be a bounded domain with smooth boundary near \( z_0 \in \partial \Omega \). Let \( U \) be a neighborhood of \( z_0 \). Let \( r(z) \) be a local defining function of \( \Omega \) on \( U \), i.e.,
\[
\Omega \cap U = \{ z \in U \mid r(z) < 0 \}
\]
and \( r(z) \in C^{\infty}(U), \nabla r(z)|_{\partial \Omega \cap U} \equiv (\partial r(z)/\partial z_1, \partial r(z)/\partial z_2, \ldots, \partial r(z)/\partial z_n) \neq 0 \).

Let \( \Delta \) be the unit disc and let \( \Delta_\gamma \equiv \{ \gamma \zeta; \zeta \in \Delta \} \). The Kobayashi metric of \( \Omega \) is defined by
\[
F_\Omega(z, X) \equiv \inf \left\{ 1/\lambda \mid \exists f: \Delta \rightarrow \Omega \text{ is holomorphic and } f(0) = z, f'(0) = \lambda X, \lambda > 0 \right\}.
\]

For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \), set \( z' = (z_1, z_2, \ldots, z_{n-1}) \). Let \( d(z) = \text{dist}(z, \partial \Omega) \) and let \( \pi(z) \) be the projection to the boundary for \( z \) near \( z_0 \) such that \( d(z) = |z - \pi(z)| \). Let \( N_{\pi(z)} \) be the inward normal direction at \( \pi(z) \). Denote
\[
H_{z_0} \equiv \left\{ X \in \mathbb{C}^n \mid (\partial r(z_0), X) \equiv \sum_{i=1}^{n} \frac{\partial r(z_0)}{\partial z_i} X_i = 0 \right\}.
\]

We call \( z_0 \in \partial \Omega \) a Levi-pseudoconvex point if
\[
\sum_{i,j=1}^{n} \frac{\partial^2 r(z_0)}{\partial z_i \partial \bar{z}_j} X_i \bar{X}_j \geq 0 \text{ for all } X \in H_{z_0}.
\]
In [K], Krantz has proven that if $\Omega$ is a bounded domain with smooth boundary near $z_0$, then there exists constant $C > 0$ such that

$$F_\Omega(z, \nabla r(z)) \geq C \frac{|\nabla r(z)|}{d^{3/4}(z)} \quad \text{for } z \in \Omega \text{ near } z_0.$$ 

In this note, we shall prove the following theorems.

**Theorem A.** Let $\Omega \subset \subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial \Omega$. Suppose that there exist $\alpha > 3/4$, $C > 0$, $X \in \mathbb{C}^n \setminus H_{z_0}$, and $\{z_k\}_{k=1}^\infty$ with $z_k \to z_0$ nontangentially (i.e., $z_k$ stays in some cone $\Lambda$ with vertex at $z_0$ and axis $N_{z_0}$) such that

$$F_\Omega(z_k, X) \geq C \frac{|\partial r(z_k)|}{d^\alpha(z_k)} \quad \text{for all } k \in \mathbb{N}.$$

Then $z_0$ is a Levi-pseudoconvex point.

**Theorem B.** Let $\Omega \subset \subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial \Omega$, and let $\Lambda$ be a cone with vertex at $z_0$ and axis $N_{z_0}$. Assume that $z_0$ is the origin. Suppose that the local defining function of $\Omega$ has the following form near $z_0$:

$$r(z) = \Re z_n + \Theta(|z'|^m + |z_n| \cdot |z|).$$

Then there exist a neighborhood $V$ of $z_0$ and a constant $C > 0$ such that

$$F_\Omega(z, X) \geq C \frac{|X_n|}{d^{1-\frac{1}{\alpha}}(z)} \quad \text{for } z \in \Lambda \cap \Omega \cap V \text{ and all } X \in \mathbb{C}^n .$$

Furthermore, there exists $C_1 > 0$ such that

$$F_\Omega(z, X) \geq C_1 \frac{|X_n|}{d^{1-\frac{1}{\alpha}}(z)} \quad \text{for } z \in \Lambda \cap \Omega \cap V \text{ and } X \in \mathbb{C}^n \text{ with } |X| \leq K|X_n| \quad (C_1 \text{ may depend on the constant } K).$$

As corollaries of Theorem A and Theorem B, we obtain

**Corollary 1.** Let $\Omega \subset \subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial \Omega$, and let $\Lambda$ be a cone with vertex at $z_0$ and axis $N_{z_0}$. Then $z_0$ is a Levi-pseudoconvex point if and only if there exist a neighborhood $V$ of $z_0$, $\alpha > 3/4$, and $C > 0$ such that

$$F_\Omega(z, \nabla r(z_0)) \geq C \frac{|\nabla r(z_0)|}{d^\alpha(z)} \quad \text{for all } z \in \Lambda \cap \Omega \cap V .$$

**Corollary 2.** Let $\Omega \subset \subset \mathbb{C}^2$ be a bounded domain with smooth pseudoconvex boundary near $z_0 \in \partial \Omega$ and let $\Lambda$ be a cone with vertex at $z_0$ and axis $N_{z_0}$. Then for each $\alpha \in (0, 1)$, there exist a neighborhood $V$ of $z_0$ and a constant $C > 0$ such that

$$F_\Omega(z, X) \geq C \frac{|(\partial r(z_0), X)|}{d^\alpha(z)} \quad \text{for all } z \in \Lambda \cap \Omega \cap V \text{ and } X \in \mathbb{C}^n .$$
II. PROOFS OF THE THEOREMS

Some ideas in the proofs of Theorem A and Theorem B come from [K]. We will also need the following lemma, which can be proven directly from Lemma 2 in [R].

**Lemma.** Let Ω' be a subdomain of a bounded domain Ω with ∂Ω' ∩ ∂Ω ⊂ U ∩ ∂Ω for some neighborhood U of z₀ ∈ ∂Ω. Then there exist a neighborhood V ⊂⊂ U of z₀ and a constant C > 0 such that

\[ F_{Ω'}(z, X) \leq C F_Ω(z, X) \]

for z ∈ Ω' ∩ V and X ∈ C^n.

We will use C to denote constants which may be different in different appearances.

**Proof of Theorem A.** After a translation and a unitary transformation, we may assume that z₀ is the origin and ∂Ω is locally defined by

\[ r(z) = \text{Re } z_n + \sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j + O(|z|^3) \]

for z near z₀.

Suppose that ∂Ω is not Levi-pseudoconvex at z₀. Then the matrix

\[ \left( \frac{\partial^2 r(z_0)}{\partial z_i \partial \bar{z}_j} \right)_{1 \leq i,j \leq n-1} \]

has at least one negative eigenvalue. Therefore, after a unitary transformation in z' = (z₁, z₂, ..., zₙ₋₁) and a simple change of coordinate system, we may assume that

\[ r(z) = \text{Re } z_n - |z_1|^2 + \sum_{i,j=2}^{n} a_{ij} z_i \bar{z}_j + O(|z|^3) \]

for z ∈ U, where U is some neighborhood of z₀. Shrinking U, we have

\[ r(z) \leq \text{Re } z_n - \frac{|z_1|^2}{2} + C \sum_{i=2}^{n} |z_i|^2, \quad \text{for } z \in U. \]

Let Λ = \{ - \text{Re } z_n > k|z| \} (0 < k < 1) be the cone. By the Implicit Function Theorem,

\[ \lim_{z \to 0} \frac{-\text{Re } z_n}{d(z)} = 1. \]

By the homogeneity of the Kobayashi metric, we may assume that X = (X₁, X₂, ..., Xₙ₋₁, 1). For z = (z', zₙ) = (z₁, z₂, ..., zₙ) ∈ Λ ∩ Ω ∩ U, let δ = -Re zₙ. Define \( \Phi_δ(ζ) = (Φ₁δ(ζ), Φ₂δ(ζ), ..., Φₙδ(ζ)) \) by

\[ Φ₁δ = z₁ + \frac{δ^{3/4}}{2} X₁ζ + 2ζ^2; \]

\[ Φ_kδ = z_k + \frac{δ^{3/4}}{2} X_kζ, \quad \text{for } 2 \leq k \leq n-1; \]

\[ Φₙδ = zₙ + \frac{δ^{3/4}}{2} ζ. \]

Then \( Φ₂(0) = z, \ Φ₂'(0) = \frac{δ^{3/4}}{2} X \).
**Claim.** There exists $\gamma \in (0, 1)$ such that for all $\delta > 0$ sufficiently small, we have $\Phi_{\delta}(\Delta \gamma) \subset \Omega \cap U$.

**Proof of the Claim.** By choosing $\gamma \in (0, 1)$ small enough, we have $\Phi_{\delta}(\Delta \gamma) \subset U$. It follows from (2.3) that

$$r(\Phi_{\delta}(\zeta)) \leq -\delta + \frac{\delta^{3/4}}{2} \text{Re} \zeta - \frac{\omega_{1\delta}(\zeta)^2}{2} + C \sum_{i=2}^n |\omega_{1i}(\zeta)|^2.$$

Since $\delta > k|\zeta|$, we see that when $\delta$ is sufficiently small,

$$r(\Phi_{\delta}(\zeta)) < -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \text{Re} \zeta - \frac{\omega_{1\delta}(\zeta)^2}{2}.$$

For $|\zeta| < \delta^{1/4}$, by (2.5),

$$r(\Phi_{\delta}(\zeta)) < -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \cdot \delta^{1/4} - \frac{\omega_{1\delta}(\zeta)^2}{2} = -\frac{\delta}{4} - \frac{\omega_{1\delta}(\zeta)^2}{2} < 0.$$

For $|\zeta| \geq \delta^{1/4}$, we have

$$|\omega_{1\delta}(\zeta)|^2 \geq 2|\zeta|^4 - |z_1 + \frac{\delta^{3/4}}{2} x_1 \zeta|^2 \geq 2|\zeta|^4 - C \delta^{3/2}.$$

Thus (2.5) implies that when $\delta$ sufficiently small,

$$r(\Phi_{\delta}(\zeta)) < -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \text{Re} \zeta - |\zeta|^4 + \frac{C \delta^{3/2}}{2} \leq -\frac{3\delta}{4} + \frac{C \delta^{3/2}}{2} + \frac{|\zeta|^4}{2} - |\zeta|^4 < 0.$$

This concludes the proof of the Claim.

Next, by the Claim and the definition of the Kobayashi metric,

$$F_{\Omega \cap U}(z, X) \leq \frac{C}{\delta^{3/4}}.$$

Thus, combining (2.4) and the length-decreasing property of Kobayashi metric, we obtain

$$F_{\Omega}(z, X) \leq \frac{C}{d^{3/4}(z)},$$

which contradicts (1.1). Therefore, $\partial \Omega$ is Levi-pseudoconvex at $z_0$. $\Box$

**Proof of Theorem B.** By the assumption, there exists a neighborhood $U$ of $z_0$ such that

$$\Omega \cap U \subset \{ z \in U \mid \text{Re} z_n - C(|z'|^m + |z_n| \cdot |z|) < 0 \}.$$

Let $\Lambda = \{- \text{Re} z_n > k|\zeta|\} \ (k \in (0, 1))$. For $z \in \Lambda \cap \Omega \cap U$ and $X \in \mathbb{C}^n$ (by homogeneity of the Kobayashi metric, we may assume that $|X| \leq 1$), let
\[ \Phi(\zeta) = (\Phi_1(\zeta), \Phi_n(\zeta)) = (\Phi_1(\zeta), \Phi_2(\zeta), \ldots, \Phi_n(\zeta)): \Delta \to \Omega \cap U \] be an analytic disc satisfying
\[ \Phi(0) = z, \quad \Phi'(0) = \lambda X, \]
where \( \lambda > 0 \) is a constant to be estimated. By the Cauchy Integral Formula, we have
\[ |\Phi_i(\zeta) - z_i| \leq C|\zeta|, \quad 1 \leq i \leq n, \] and
\[ |\Phi_i(\zeta) - z_i - \lambda X_i \zeta| \leq C|\zeta|^2, \quad 1 \leq i \leq n, \]
for \( |\zeta| < 1/2 \). Also, by (2.6), we have
\[ \text{Re} \Phi_n(\zeta) < C \left( |\Phi(\zeta)|^m + |\Phi_n(\zeta)| \cdot |\Phi(\zeta)| \right). \]
Denote \( \delta = -\text{Re} \ z_n \). By (2.8), (2.8'), and the fact that \( k|z| < \delta \), we have
\[ |\Phi(\zeta)| \leq C(|z| + |\zeta|) \]
and
\[ |\Phi(\zeta)| \leq C \left( |z| + (\lambda|X|)|\zeta| + |\zeta|^2 \right) \]
\[ \leq C \left( (1/k)\delta + c\delta^{1/m} \right) \]
\[ \leq c^{1/2}\delta^{1/m}, \quad \text{for} \quad |\zeta| < c\delta^{1/m} \]
when \( c, \delta \) are sufficiently small.
Now, it follows from (2.10), (2.10'), (2.9) that
\[ \text{Re} \Phi_n(\zeta) < \frac{\delta}{2} + \frac{1}{2}|\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/m} \]
and
\[ \text{Re} \Phi_n(\zeta) < \frac{\delta + (\lambda|X|)^m\delta^{1/2}}{2} + \frac{1}{2}|\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/2m} \]
when \( c, \delta \) are sufficiently small.
Denote
\[ D_{i\delta} = \left\{ w \in \mathbb{C} \mid \text{Re} \ w < \frac{\delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2}}{2} + \frac{1}{2}|w| \right\}, \quad \text{for} \quad i = 1, 2 \]
where \( \varepsilon_1 = 0, \varepsilon_2 = 1 \). Let \( g_1(\zeta) \equiv \Phi_n(c\delta^{1/m}\zeta) \) and \( g_2(\zeta) \equiv \Phi_n(c\delta^{1/2m}\zeta) \).
By (2.7), (2.11), and (2.11'), we have \( g_i(\Delta) \subset D_{i\delta} \), \( g_i(0) = z_n \) (\( i = 1, 2 \)), \( g_1'(0) = \lambda X_n c\delta^{1/m} \), and \( g_2'(0) = \lambda X_n c\delta^{1/2m} \). However, it is clear that
\[ D_{i\delta} \subset \bar{D}_{i\delta} \equiv \mathbb{C} \setminus \left\{ w \in \mathbb{C} \mid \text{Im} \ w = 0, \text{Re} \ w \geq \delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2} \right\}. \]
Thus \( g_i(\Delta) \subset \bar{D}_{i\delta} \).
Since
\[
F_{\delta}(z_n, 1) \geq \frac{C}{\delta + \varepsilon(\lambda|X|^m)\delta^{1/2}}
\]
where \( C > 0 \) is a constant independent on \( \delta \), it follows that
\[
|g'_\lambda(0)| \leq C(\delta + \varepsilon(\lambda|X|^m)\delta^{1/2}).
\]
Therefore,
\[
\lambda|X_n|\delta^{1/m} \leq C\delta
\]
and
\[
\lambda|X_n|\delta^{1/2m} \leq C(\delta + (\lambda|X|^m)\delta^{1/2}).
\]

Now by (2.12), \( \lambda|X_n| \leq C\delta^{1-1/m} \). Thus (1.2) is valid. Furthermore, when \( |X| < K|X_n| \), it follows from (2.13) that \( \lambda|X_n| \leq C\delta^{1-1/2m} \) where \( C > 0 \) may depend on \( K \). Therefore, (1.3) follows. \( \Box \)

Proof of Corollary 1. The sufficiency comes directly from Theorem A. We now prove the necessity. Suppose that \( z_0 \) is a Levi-pseudoconvex point. After a change of coordinates, we may assume that \( z_0 \) is the origin and \( \partial\Omega \) is locally defined by
\[
r(z) = \Re z_n + \sum_{i,j=1}^{n} a_{ij} z_i z_j + \mathcal{O}(|z|^3)
\]
near \( z_0 \).

Since the matrix \( (a_{ij})_{1 \leq i,j \leq n-1} \) is positive semidefinite,
\[
r(z) \geq \Re z_n + 2 \Re \left( \sum_{j=1}^{n-1} a_{nj} z_n z_j \right) + a_{nn}|z_n|^2 + \mathcal{O}(|z|^3)
\]
\[
= \Re z_n + \mathcal{O}(|z'|^3 + |z_n| \cdot |z|).
\]
Therefore, (1.4) is valid for \( \alpha = 5/6 \) by Theorem B. \( \Box \)

Proof of Corollary 2. When \( z_0 \) is of finite type, as in Property 2.1 in [M], we can construct a smoothly bounded pseudoconvex domain \( \Omega' \subset \Omega \) such that \( \partial\Omega' \cap \partial\Omega \supset \partial\Omega \cap U \) for some neighborhood \( U \) of \( z_0 \). By the Lemma, there exist a neighborhood \( V \) of \( z_0 \) and a constant \( C > 0 \) such that
\[
F_{\Omega}(z, X) \geq CF_{\Omega'}(z, X)
\]
for \( z \in V \cap \Omega', X \in \mathbb{C}^n \). Applying Theorem 1 in [C] to \( \Omega' \), we then obtain that (1.5) is valid for \( \alpha = 1 \).

When \( z_0 \) is of infinite type, then for each \( m \in \mathbb{N} \), after a change of coordinates, the local defining function of \( \Omega \) near \( z_0 \) has the form
\[
r(z) = \Re z_2 + \mathcal{O}(|z_1|^m + |z_2| \cdot |z|).
\]
Thus, (1.5) follows from Theorem B. \( \Box \)

Remark. The estimates in Theorem B are sharp. Suppose that \( \Omega \) is a bounded domain such that \( \partial\Omega \) is locally defined by \( r(z) = \Re z_2 - |z_1|^{2m} \) near the origin. Then for \( \delta = (0, -\delta) \), \( X = (\delta^{-1+1/2m}, 1) \), and \( Y = (0, 1) \), we have
\[
F_{\Omega}(\delta, X) \approx \frac{1}{\delta^{1-1/2m}}, \quad F_{\Omega}(\delta, Y) \approx \frac{1}{\delta^{1-1/4m}}.
\]
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