

DETECTING NEGATIVE CURVATURE IN GROUPS VIA LOCAL CONDITIONS

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ABSTRACT. We characterize negative curvature of groups in terms of properties which can be checked locally in the Cayley graph. We prove the equivalence of these properties with the more familiar thin triangles property. We also present an algorithm which, when given a finite presentation of a group G , will recognize these properties in finite time if G is negatively curved.

0. INTRODUCTION

In this paper we present a new characterization of negatively curved or Gromov-hyperbolic groups. We prove that this new characterization is equivalent to the more familiar definition which is stated in terms of thin triangles. We also show that this new characterization leads to a partial algorithm for verifying negative curvature in groups.

The distinctive feature of our new characterization is that we can verify it by looking at neighborhoods of finitely many vertices of the Cayley graph which arises from some finite presentation. It is known that every negatively curved group admits a finite presentation, thus we restrict ourselves to that category. See [Gr, 1.7 and 2.2], [GdlH, Proposition 4.17]. We refer to the components of our new characterization as Local Conditions.

0.1. Setting. Let $G = \langle A | R \rangle$ be a group with a fixed finite presentation. Let Γ be the Cayley graph of G with respect to this presentation. We consider each edge of Γ to be the isometric image of $[0, 1]$, and thus (Γ, d) is a connected geodesic metric space: given any two points there exists a path joining them which realizes the distance between them. We will let $[x, y]$ denote a (possibly nonunique) geodesic segment joining the points $x, y \in \Gamma$.

We say that G is *negatively curved* or *Gromov-hyperbolic* if geodesic triangles in Γ are uniformly thin: there exists a number $\delta \geq 0$ such that given any geodesic triangle, a point on one side is within δ of some point in the union of the other two sides. We refer to this condition by saying that G has thin triangles (δ) ; if there is no need to specify δ , we say simply that G has thin triangles. We may take δ to be a nonnegative integer.

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Negative curvature is a property of the group and is independent of presentation. If two distinct finite presentations give rise to isomorphic groups, then the corresponding Cayley graphs both have thin triangles, though possibly with different values of δ in each. See [GdlH, Corollary 5.14], [Ge], [Gh]. For more discussion on negatively curved groups see [B], [C], [ABC].

For any path P in Γ let $\ell(P)$ denote the length of P . Let O denote a fixed vertex of Γ , and for $n \geq 0$, let $B(n)$, $R(n)$, and $S(n)$ denote the closed ball, open ball, and sphere of radius n centered at O . Let $A[m, n]$ denote the annulus of inner radius m and outer radius n . For an arbitrary vertex y , let $B(y, n)$ denote the ball of radius n , or n -ball, centered at y . Precise definitions are as follows:

$$\begin{aligned} B(n) &= \{x \in \Gamma \mid d(O, x) \leq n\}, \\ R(n) &= \{x \in \Gamma \mid d(O, x) < n\}, \\ S(n) &= \{x \in \Gamma \mid d(O, x) = n\}, \\ A[m, n] &= \{x \in \Gamma \mid m \leq d(O, x) \leq n\}, \\ B(y, n) &= \{x \in \Gamma \mid d(x, y) \leq n\}. \end{aligned}$$

1. CONDITIONS

1.1. **Definitions and conditions.** We begin by defining radial geodesic and radial quasi-geodesic.

Definition 1.1. An edge path $V \subset \Gamma$ is a *radial geodesic* if for any edge path $P \subset V$ with endpoints p_1 and p_2 , $\ell(P) = |d(O, p_1) - d(O, p_2)|$.

Definition 1.2. Let j and j' be positive integers. An edge path U is a *radial (j, j') -quasi-geodesic* if for any edge path $P \subset U$ with endpoints p_1 and p_2 , if $\ell(P) \geq j$, then $\ell(P) \leq j'|d(O, p_1) - d(O, p_2)|$.

Remark 1.3. A radial geodesic is a radial $(j, 1)$ -quasi-geodesic for any j . A radial quasi-geodesic varies only slightly from a true radial geodesic. The variable j' gives a bound on the variation, while j tells us how long of a subsegment we need in order to detect this variation.

Remark 1.4. If U is a radial (j, j') -quasi-geodesic with endpoints u_1 and u_2 and if $\ell(U) \geq j$, then it follows from Definition 2 that $|d(O, u_1) - d(O, u_2)| \geq \frac{j}{j'}$. In particular, $d(O, u_1) \neq d(O, u_2)$. If $d(O, u_1) > d(O, u_2)$, then we say that u_1 is the *outer endpoint* of U .

We will show that the following conditions are equivalent to thin triangles.

The local conditions. There exist integers $k \geq 2$ and $l \geq 2k$ such that the following hold. (See Figures 1.1(a)-(c).)

(1) Let P be an edge path in $A[n, n+k]$ for some n . If P has one endpoint in $S(n)$ and the other in $S(n) \cup S(n+1)$ and if P leaves $B(n+k-1)$, then there exists an edge path P' with the same endpoints as P and $\ell(P') < \ell(P)$.

(2) Let P be an edge path in $A[n, n+k]$ for some integer $n \geq 0$. If $\ell(P) = l+1$, then there exists an edge path P' with the same endpoints as P such that $\ell(P') < \ell(P)$.

(3) Let U be a geodesic edge path with endpoints x and z such that $d(O, x) - d(O, z) = 4k$. Let $V = [O, y]$ with outer endpoint y . Assume

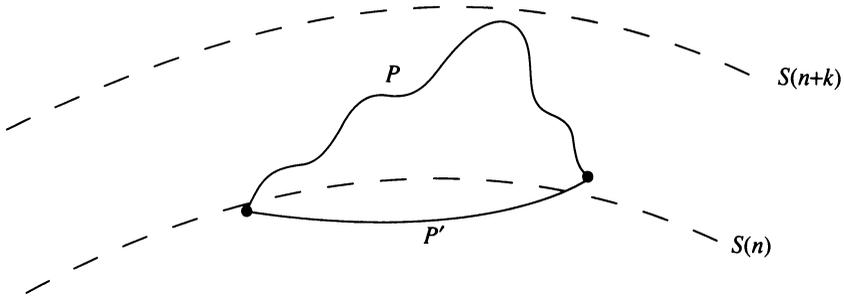


FIGURE 1.1(a)

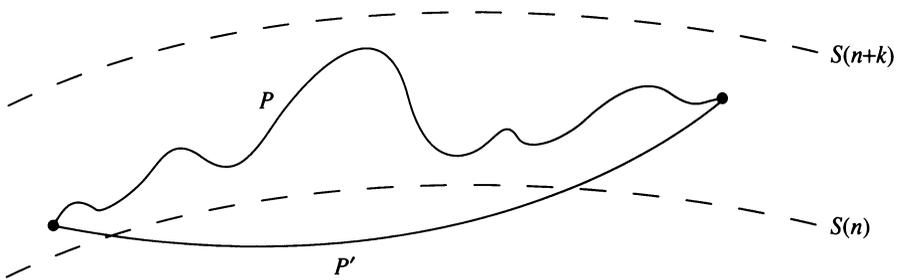


FIGURE 1.1(b)

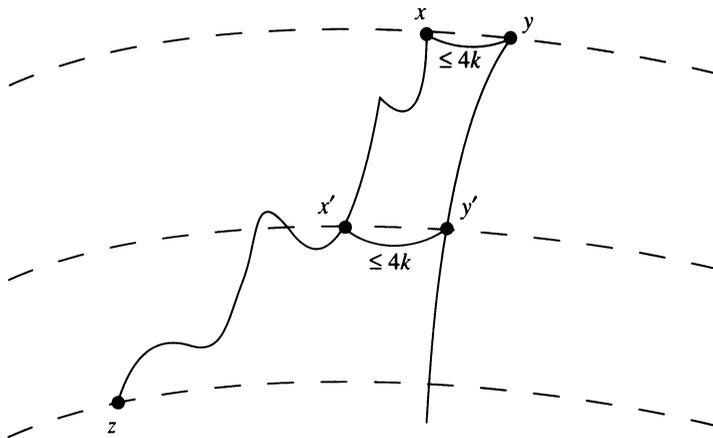


FIGURE 1.1(c)

$d(O, x) = d(O, y)$. Order the points of U from z to x and let x' be the maximal vertex of $U \cap S(d(O, x) - 2k)$. Let y' be the unique vertex in $V \cap S(d(O, y) - 2k)$. If $d(x, y) \leq 4k$, then $d(x', y') \leq 4k$.

1.2. Verifying the conditions. Assume the local conditions hold in Γ . We outline a way to verify them. First, we make precise what it means to verify each condition *at a vertex* y .

To check condition (1) at y we look at all edge paths of length at most l

that have y as an endpoint and that reach out and back in the way described in condition (1), verifying that such paths are not geodesics.

To check condition (2) we look at all edge paths of length $l + 1$ having one endpoint at y and lying in some annulus of width k , verifying that such paths are not geodesics.

We verify condition (3) at y as follows. Let x be a fixed vertex on the same level as y and such that $d(x, y) \leq 4k$. Let $U = [z, x]$ be a geodesic with $d(O, x) - d(O, z) = 4k$ and with the points ordered from z to x . Let z' be the maximal vertex of $U \cap S(d(O, z))$. An application of Theorem 2.1, given below, shows that $\ell([z', x]) \leq 4kl$. That subsegment is the only part of U we need, so we may as well assume that $\ell(U) \leq 4kl$. Let V be a radial geodesic segment of length at least $2k$ with outer endpoint y . We verify condition (3) on this U and V .

Now we let x and z vary over all vertices such that $d(O, y) = d(O, x)$, $d(O, x) - d(O, z) = 4k$, and $d(x, z) \leq 4kl$. We also let V vary over other radial geodesic segments with outer endpoint y .

We now see the local nature of these conditions. A neighborhood of radius $4k + 4kl$ contains everything we need to see in order to verify the conditions. Therefore, we set $N = 4k + 4kl$ and we classify the vertices of Γ according to their N -types. This classification is made precise as follows. Given two vertices $x, y \in \Gamma$ we say that x and y are equivalent (have the same N -type) if the graph automorphism $T : \Gamma \rightarrow \Gamma$ taking x to y by left multiplication by yx^{-1} has the property that

$$d(O, x') - d(O, x) = d(O, Tx') - d(O, y)$$

for each $x' \in B(x, N)$. In simpler terms, $x \sim y$ if the restricted map $T|B(x, N) : B(x, N) \rightarrow B(y, N)$ preserves distances from the origin, relative to x and y .

Since G is finitely presented, there are only finitely many equivalence classes (N -types) of vertices. To verify the local conditions in Γ , we choose a vertex from each equivalence class and check the conditions at those vertices, as described above.

2. CONDITIONS IMPLY NEGATIVE CURVATURE

In this section we prove that if the local conditions hold in Γ , then Γ has thin triangles. Let O denote a fixed vertex in Γ .

Conditions (1) and (2) have consequences for the structure of long geodesic segments in Γ . Condition (1) says that geodesics can have no tall peaks. Suppose P is an edge path with its endpoints in $S(n)$ for some integer n and with another point z outside of $R(n + k)$. A consequence of condition (1) is that such a path is not geodesic.

Condition (2) says that if $x, y \in A[n, n + k]$ and if a geodesic path P joining them is long, then P must leave $A[n, n + k]$. Condition (1) ensures that P dips inside of $R(n)$. Thus, long geodesics tend towards the origin, an expected behavior in negatively curved spaces. In fact, Theorem 2.1 shows that P consists of a segment P_0 of bounded length containing the innermost points of P and two segments reaching almost directly outward from the endpoints of P_0 . Theorem 2.1 shows these segments are radial quasi-geodesics. Theorem

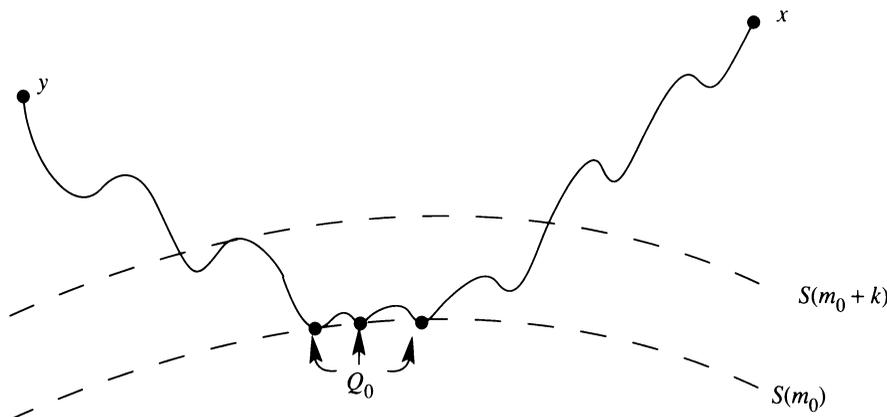


FIGURE 2.1

2.2 uses condition (3) to show that these radial quasi-geodesics are close to the respective sides $[O, x]$ and $[O, y]$. It follows that the triangle Oxy is thin.

Theorem 2.1. *Suppose conditions (1) and (2) hold in Γ . Let $O \in \Gamma$ denote a fixed origin. Let x and y be two distinct vertices in Γ , and let $[x, y]$ denote a geodesic joining them. Then there exist two vertices x_0 and y_0 on $[x, y]$ such that*

- (1) *the length of the subsegment $[x_0, y_0]$ is bounded by $l - 2k$, and*
- (2) *the segments $[x, x_0]$ and $[y_0, y]$ are radial (kl, l) -quasi-geodesics.*

Proof. The cases where $x = O$ or $y = O$ are trivial. Assume from now on that $x \neq O$ and $y \neq O$.

Let Q_0 be the set of innermost vertices in $[x, y]$ and let $m_0 = d(O, Q)$. If $m_0 = 0$, then the theorem holds with $x_0 = y_0 = O$. Now suppose that $m_0 > 0$. I claim that Q_0 is contained in a connected component of $A[m_0, m_0 + k] \cap [x, y]$. (See Figure 2.1.) Suppose this is false. Then there are two vertices $a, b \in Q_0$ which are in two distinct components of $A[m_0, m_0 + k] \cap [x, y]$, and there is a point $z \in [x, y]$ which separates a and b for which $d(O, z) > m_0 + k$. By condition (1) we can shorten $[x, y]$, but this contradicts our hypothesis that $[x, y]$ is geodesic.

Let P_0 denote the component of $A[m_0, m_0 + k] \cap [x, y]$ which contains Q_0 . Let \preceq denote the ordering of the points of $[x, y]$ from y to x . Let y_0 and x_0 be the minimal and maximal vertices of Q_0 . By condition (2), $\ell(P_0) \leq l$. Thus, $\ell([x_0, y_0]) \leq l - 2k$.

If $x_0 = x$ and $y_0 = y$, then we are done with the theorem. Assume then that $x_0 \neq x$. We will now prove that the segment $[x_0, x]$ is a radial (kl, l) -quasi-geodesic. If necessary, we can use the same argument to show that $[y_0, y]$ is also a radial (kl, l) -quasi-geodesic.

We must consider an arbitrary subedgepath of $[x_0, x]$ bounded by vertices r_0 and s , say, such that $x_0 \preceq r_0 \prec s \preceq x$ and $\ell([r_0, s]) \geq kl$. We construct a subdivision $\{r_0, r_1, \dots, r_n = s\}$ of $[r_0, s]$ as follows.

Let Q_1 be the set of innermost vertices in $[r_0, s]$ and set $m_1 = d(O, Q_1)$. We make the following claims:

- (i) $m_0 \leq m_1$.
- (ii) $m_1 \leq d(O, r_0) \leq m_1 + k - 2$.

- (iii) $Q_1 \cup \{r_0\}$ is contained in a connected component of $A[m_1, m_1 + k] \cap [r_0, s]$.

Statement (i) is clear. The case $m_0 = m_1$ can happen only when $r_0 = x_0$, which implies that $Q_1 = \{r_0\}$. Statements (ii) and (iii) are then obvious.

Consider the case where $m_0 < m_1$. If (ii) fails, then since r_0 is a vertex we must have $d(O, r_0) \geq m_1 + k - 1$. Let $a \in Q_1$, and let b be a vertex in $[x_0, r_0]$ for which $d(O, b) = m_1 - 1$. The current case of item (i) assures us that such a vertex b exists. The segment $[a, b]$, or some subsegment, can be shortened by condition (1), a contradiction.

Now for item (iii). By the same argument used above on Q_0 , all vertices of Q_1 are in the same component of $A[m_1, m_1 + k] \cap [r_0, s]$. Suppose therefore that r_0 is separated from Q_1 by a point z with $d(O, z) > m_1 + k$. Then an application of condition (1) allows us to shorten our geodesic segment. Thus, (iii) holds.

Let P_1 be the component of $A[m_1, m_1 + k] \cap [r_0, s]$ which contains r_0 and Q_1 . One endpoint of P_1 is r_0 . Let r_1 denote the other endpoint. Then $\ell([r_0, r_1]) \leq l$ by condition (2). Since $\ell([r_0, s]) \geq kl$ and $k \geq 2$ we see that $r_1 \neq s$. Therefore, $r_1 \in S(m_1 + k)$. (See Figure 2.2.)

We proceed inductively. Having defined r_{i-1} and P_{i-1} , we let Q_i denote the set of innermost vertices of $[r_{i-1}, s]$ and set $m_i = d(O, Q_i)$. Then, as above, we make the following claims:

- (i) $m_{i-1} < m_i$.
- (ii) $m_i \leq d(O, r_{i-1}) \leq m_i + k - 2$.
- (iii) $Q_i \cup \{r_{i-1}\}$ is in a connected component of $A[m_i, m_i + k - 1] \cap [r_{i-1}, s]$.

We only need to prove (i). Items (ii) and (iii) follow as above. Suppose $m_{i-1} \geq m_i$. Let $b' \in Q_i$. Let $b \in [r_{i-1}, b']$ be a vertex such that $d(O, b) = m_{i-1}$. Let $a \in [x_0, r_{i-1}]$ be a vertex such that $d(O, a) = d(O, b)$. Note that $d(O, r_{i-1}) = m_{i-1} + k$. Thus, some subsegment of $[a, b]$ can be shortened by condition (1).

We define P_i and r_i in the expected way. As before, $\ell(P_i) \leq l$. If $r_i \neq s$, then we note that $r_i \in S(m_i + k)$ and we proceed with the induction.

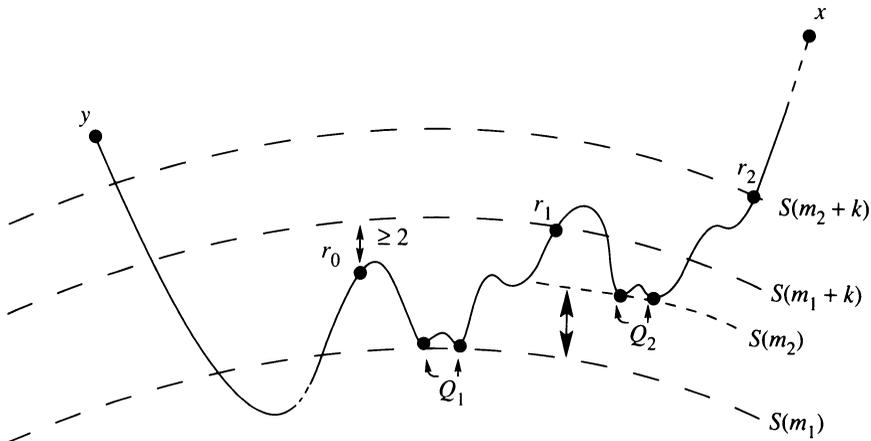


FIGURE 2.2

Eventually, we arrive at $r_n = s$ for some $n \geq k$. For $i = 1, 2, \dots, n - 1$,

$$\begin{aligned} d(O, r_i) - d(O, r_{i-1}) &= (d(O, r_i) - m_i) - (d(O, r_{i-1}) - m_i) \\ &\geq k - (k - 2) \geq 2 \end{aligned}$$

and

$$d(O, r_n) - d(O, r_{n-1}) \geq -(k - 2).$$

Thus,

$$\begin{aligned} d(O, s) - d(O, r_0) &= d(O, r_n) - d(O, r_0) = \sum_{i=1}^n (d(O, r_i) - d(O, r_{i-1})) \\ &\geq 2(n - 1) - (k - 2) = 2n - k \geq n. \end{aligned}$$

We obtain

$$\ell([r_0, s]) \leq \sum_{i=1}^n \ell(P_i) \leq nl \leq l|d(O, s) - d(O, r_0)|. \quad \square$$

Theorem 2.2. *If Γ satisfies the local conditions, then Γ has thin triangles (δ) for some $\delta \geq 0$, and hence the group G is negatively curved.*

Proof. Suppose given a geodesic triangle in Γ . Since G is a group, Γ is a homogeneous metric space. We can choose one of the vertices of the triangle to be the identity vertex O and call the other vertices x and y .

By Theorem 2.1, there are two vertices $x_0, y_0 \in [x, y]$ such that $\ell([x_0, y_0]) \leq l - 2k$ and the segments $[x_0, x]$ and $[y_0, y]$ are radial (kl, l) -quasi-geodesics.

If $d(O, x) - d(O, x_0) \leq 4k$, then $d(x_0, x) \leq 4kl$ by Definition 1.2. It follows that each point of $[x, x_0]$ is within $4kl$ of some point of $[O, x]$.

Suppose on the other hand that $d(O, x) - d(O, x_0) \geq 4k$. Set $m = d(O, x)$. Order the points of $[x_0, x]$ from x_0 to x . We subdivide $[x_0, x]$ as follows. For $i = 1, 2, \dots, n + 1$ let x_i be the maximal vertex of $[x_0, x] \cap S(m - 2ik)$, where $n + 1$ is the largest integer such that $[x_0, x] \cap S(m - 2(n + 1)k)$ is nonempty. For each $i = 1, 2, \dots, n$ let z_i be the unique vertex on $[O, x]$ such that $d(O, x_i) = d(O, z_i)$. By condition (3), $d(x_i, z_i) \leq 4k$. Note that $d(O, x_{i-1}) - d(O, x_i) = 2k$, so by Definition 1.2, $d(x_{i-1}, x_i) \leq 2kl$. Each point of $[x_n, x]$ is within $4k + kl$ of some point of $[O, x]$. We make a similar subdivision $\{y_1, \dots, y_{n'}\}$ of $[y_0, y]$. Observe that each point of $[y_{n'}, y]$ is within $4k + kl$ of some point of $[O, y]$. (See Figure 2.3 on the next page.)

We compute a bound on the length of $[y_{n'}, x_n]$. Since $d(O, x_n) - d(O, x_0) \leq 4k$, Definition 1.2 implies

$$\ell([x_0, x_n]) \leq 4kl$$

and similarly

$$\ell([y_{n'}, y_0]) \leq 4kl.$$

Recall from the proof of Theorem 3.1 that $\ell([x_0, y_0]) \leq l - 2k$. The bound we seek is $s = l - 2k + 8kl$ and thus, any point of $[x, y]$ is within $4k + \max\{\frac{1}{2}s, kl\}$.

We permute the labels on the vertices and repeat the argument. It follows that Γ has thin triangles (δ) where $\delta = 4k + \max\{\frac{1}{2}s, kl\}$. \square

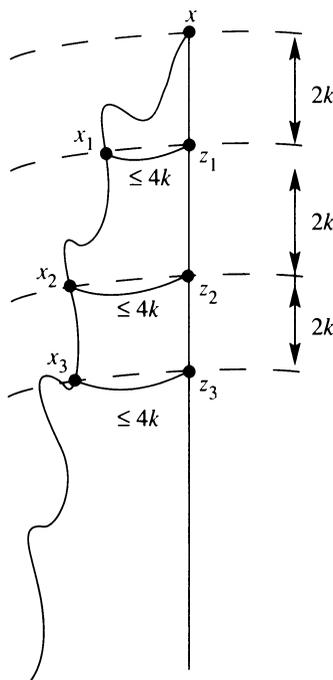


FIGURE 2.3

3. NEGATIVE CURVATURE IMPLIES LOCAL CONDITIONS

In this section we complete the proof of the equivalence of the local conditions and the thin triangles property.

Theorem 3.1. *If Γ has thin triangles (δ) for some $\delta \geq 0$, then Γ satisfies the local conditions.*

Sketch of proof. We may take δ to be an integer ≥ 0 . Set $k = \delta + 2$ and $l = 2k + 4\delta$. With these values of k and l , we prove that each local condition holds. The details of the proof are left to the reader. \square

4. AN ALGORITHM FOR DETECTING NEGATIVE CURVATURE

Suppose we have a finitely presented group $G = \langle A | R \rangle$ that we suspect is negatively curved. There is a partial algorithm which will verify this. The algorithm runs as follows.

- (1) Set $\delta = 0$.
- (2) Set $k = 2 + \delta$ and $l = 2k + 4\delta$.
- (3) Set $N = 4kl + 4k$ and enumerate all N -types in Γ .
- (4) Check the local conditions in each N -type. If they hold, then we are done. If not, increment δ by 1 and return to step (2).

If G is negatively curved, then this process will eventually terminate; otherwise, it will run forever. This is the best kind of algorithm we can hope for since negative curvature is a Markov property [LS, p. 192]

The only step not discussed so far is the enumeration of N -types. This is made reasonably easy by the fact that negatively curved groups are automatic

[ECH, Sec. 3.4]. In fact, they are shortlex automatic and this fact is independent of the presentation. Furthermore, computer programs exist which can explicitly compute the shortlex automatic structure [EHR]. This structure makes it possible to efficiently build segments of the Cayley graph. It also enables us to find a number R whose value is independent of N such that all N -types can be found in the ball of radius $R + 2N$. See [S]. We construct this ball and enumerate all N -types we find in it.

If G is not negatively curved, then it may or may not have a shortlex automatic structure. If not, the programs designed to find it will never terminate. If a shortlex automatic structure is found, we can then proceed with the algorithm above, but it will never terminate.

Algorithms for detecting negative curvature are also discussed by Papasoglu [P] and Gromov [Gr, 2.3F].

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