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ALL AUTOHOMEOMORPHISMS OF CONNECTED MENGER MANIFOLDS ARE STABLE

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ABSTRACT. Generalizing the result of Chigogidze (1991), we prove that *all* autohomeomorphisms of connected Menger manifolds are stable (in the sense of Brown and Gluck).

A manifold modeled on the $(n+1)$-dimensional universal Menger compactum μ^{n+1} is called a μ^{n+1}*-manifold.* In [C_1], Chigogidze proved that an autohomeomorphism h of a μ^{n+1}-manifold is stable (in the sense of Brown and Gluck [BG]) if h *is properly n-homotopic to* id. Here we show that *all* autohomeomorphisms of connected μ^{n+1}-manifolds are stable. Indeed the following can be proved:

Theorem. *Every autohomeomorphism h of a connected μ^{n+1}-manifold M can be written as the composition of two homeomorphisms, each of which is identity on some nonempty open set.*

The connectedness is necessary. In fact, let $M = M_1 \oplus M_2$, where $M_1 \approx M_2$ are connected.[1] Then any autohomeomorphism h of M is unstable if $h(M_1) = M_2$.

A pair $(W, \delta W)$ of compacta in M is called a *clean* μ^{n+1}*-pair* if

(1) $W \approx \delta W \approx \mu^{n+1}$,
(2) $(M \setminus W) \cup \delta W$ is a μ^{n+1}-manifold,
(3) δW is a Z-set in both W and $(M \setminus W) \cup \delta W$, and
(4) $W \setminus \delta W$ is open in M (i.e., $\mathrm{bd}_M W \subset \delta W$).[2]

Lemma. *Let $(W, \delta W)$ be a clean μ^{n+1}-pair in a μ^{n+1}-manifold M. Then $(M \setminus W) \cup \delta W \approx M$.*

Proof. First note that the inclusion $j: (M \setminus W) \cup \delta W \subset M$ induces a homeomorphism between the spaces of ends, whence j induces an isomorphism of homotopy groups of ends. By [Be, §6, Theorem], it suffices to show that j induces an isomorphism of homotopy groups of $\dim \leqslant n$.

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[1]Here " $\approx$ " means "is homeomorphic to".
[2]The existence of such a pair is guaranteed by [IS, Lemma 1] (cf. [C_2]).

Epi: For each map $f: \mathbf{S}^i \to M$ ($i \leqslant n$) of the i-sphere, let

$$A = \operatorname{cl} f^{-1}(W \setminus \delta W) \quad \text{and} \quad B = \operatorname{bd} A.$$

By [Hu, Chapter V, Theorem 10.1], $f|B$ extends to a map $g': A \to \delta W$ and there is a homotopy $h': A \times \mathbf{I} \to W$ such that $h'_0 = f|A$, $h'_1 = g'$, and $h'_t|B = f|B$ for each $t \in \mathbf{I}$. We can extend g' and h' to $g: \mathbf{S}^i \to (M \setminus W) \cup \delta W$ and $h: \mathbf{S}^i \times \mathbf{I} \to M$ by $g|\mathbf{S}^i \setminus A = h_t|\mathbf{S}^i \setminus A = f|\mathbf{S}^i \setminus A$. Then $h_0 = f$ and $h_1 = g$. This means that j induces an epimorphism of homotopy groups of $\dim \leqslant n$.

Mono: Suppose that a map $g: \mathbf{S}^i \to (M \setminus W) \cup \delta W$ ($i \leqslant n$) extends to a map $f: \mathbf{B}^{i+1} \to M$ of the $(i+1)$-ball. Let $C = \operatorname{cl} f^{-1}(W \setminus \delta W)$ and $D = \operatorname{bd} C$. Similarly as above, $f|D$ extends to a map $h': C \to \delta W$. We can extend h' to $h: \mathbf{B}^{i+1} \to (M \setminus W) \cup \delta W$ by $h|\mathbf{B}^{i+1} \setminus C = f|\mathbf{B}^{i+1} \setminus C$. Since $\mathbf{S}^i \subset (\mathbf{B}^{i+1} \setminus C) \cup D$, $h|\mathbf{S}^i = f|\mathbf{S}^i = g$. This implies that j induces a monomorphism of homotopy groups of $\dim \leqslant n$. □

Proof of Theorem. Let $(W_0, \delta W_0)$ be a clean μ^{n+1}-pair in M and $a \in M$. Since M is connected, by the Z-set Unknotting Theorem [Be, §6], we have an autohomeomorphism f of M such that $f(a), f(h(a)) \in W_0 \setminus \delta W_0$. Then $(W, \delta W) = (f^{-1}(W_0), f^{-1}(\delta W_0))$ is a clean μ^{n+1}-pair and $a, h(a) \in W \setminus \delta W$. By [IS, Lemma 1], we can choose a clean μ^{n+1}-pair $(V, \delta V)$ so that $a \in V \setminus \delta V$ and $V \cup h(V) \subset W \setminus \delta W$. Note that $(h(V), h(\delta V))$ is also a clean μ^{n+1}-pair. By Lemma, $N_1 = (W \setminus V) \cup \delta V \approx \mu^{n+1}$ and $N_2 = (W \setminus h(V)) \cup h(\delta V) \approx \mu^{n+1}$. Observe that δW and δV are disjoint Z-sets in N_1, and δW and $h(\delta V)$ are disjoint Z-sets in N_2. By using the Z-set Unknotting Theorem [Be, 3.1.5], we have a homeomorphism $g: N_1 \to N_2$ such that $g|\delta W = \mathrm{id}$ and $g|\delta V = h|\delta V$, which can be extended to a homeomorphism $h_1: M \to M$ by $h_1|M \setminus W = \mathrm{id}$ and $h_1|V = h|V$. Then $h = h_2 \circ h_1$, where $h_2 = h \circ h_1^{-1}: M \to M$ is a homeomorphism such that $h_2|V = \mathrm{id}$. □

Remark. In [Wo], [CM], and [Mc], the stability of all autohomeomorphisms is shown for connected manifolds modeled on the Hilbert cube Q and a normed linear space $E \approx E^\omega$ or $\approx E^\omega_f$. Our approach is valid for these manifolds and also for manifolds modeled on the direct limits $\mathbb{R}^\infty = \operatorname{dir\,lim} \mathbb{R}^n$ and $Q^\infty = \operatorname{dir\,lim} Q^n$. For required techniques, refer to [Ch], [BP], and [$S_{1,2}$].

References

[BP] C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, Monograf. Mat., vol. 58, Polish Sci. Publ., Warsaw, 1975.

[Be] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc. (no.380) **71** (1988).

[BG] M. Brown and H. Gluck, *Stable structures on manifolds*: I, II, III, Ann. of Math. (2) **79** (1964), 1–58.

[Ch] T. A. Chapman, *Lectures on Hilbert cube manifolds*, CBMS Regional Conf. Ser. in Math., vol. 28, Amer. Math. Soc., Providence, RI, 1976.

[C_1] A. Chigogidze, *Autohomeomorphisms of the universal Menger compacta are stable*, Soobshch Akad. Nauk Gruz. SSR **142 (3)** (1991), 477–479.

[C_2] ——, *Finding boundaries for Menger manifolds*, Proc. Amer. Math. Soc. **121** (1994), 631–640.

[CM] D. Curtis and R. A. McCoy, *Stable homeomorphisms on infinite-dimensional normed linear spaces*, Proc. Amer. Math. Soc. **28** (1971), 496–500.

[Hu] S.-T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, MI, 1965.

[IS] Y. Iwamoto and K. Sakai, *A mapping theorem for Q-manifolds and μ^{n+1}-manifolds*, preprint.

[Mc] R. A. McCoy, *Groups of homeomorphisms of normed linear spaces*, Pacific J. Math. **39** (1971), 735–743.

[S1] K. Sakai, *On $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds*, Topology Appl. **18** (1984), 69–80.

[S2] ——, *On $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds*, II: *Infinite deficiency*, Tsukuba J. Math. **8** (1984), 101–118.

[Wo] R. Y. T. Wong, *A note on stable homeomorphisms of infinite-dimensional manifolds*, Proc. Amer. Math. Soc. **28** (1971), 271–272.

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