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STRONG INCOMPACTNESS FOR SOME NON-PERFECT RINGS

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ABSTRACT. Answering a question of Eklof and Mekler, for each regular uncountable cardinal κ, we construct a non-left-perfect ring R_κ and a non-projective strongly κ-free left ideal I_κ such that $\mathrm{gen}(I_\kappa) = \kappa$. Moreover, if $\kappa > \aleph_1$, then I_κ is not κ-free. As consequences, we obtain results concerning incompactness spectra of non-perfect rings.

A famous result of Kaplansky says that the structure of projective modules over an arbitrary associative ring with unit reduces to the description of the countably generated ones. This is in sharp contrast with the structure of almost free modules. Given a non-left-perfect ring R and an almost free module M, the structure of M essentially depends on the combinatorial properties of $\mathrm{gen}(M)$, the minimal cardinality of an R-generating subset of M. Here, the main split goes between singular and regular cardinals. For its study, two important notions were introduced in [EM]:

M is *κ-free* provided there is a set, $\mathscr{C}$, of free submodules of M such that: (1) $\mathrm{gen}(X) < \kappa$ for all $X \in \mathscr{C}$, (2) for each subset $A \subseteq M$ with $\mathrm{card}(A) < \kappa$ there is some $X \in \mathscr{C}$ with $A \subseteq X$, and (3) $\mathscr{C}$ is closed under unions of well-ordered chains of length $< \kappa$.

M is *strongly κ-free* provided there is a set, $\mathscr{S}$, of free submodules of M such that: (1) $\mathrm{gen}(X) < \kappa$ for all $X \in \mathscr{S}$, (2) $0 \in \mathscr{S}$, and (3) for any subset $A \subseteq M$ with $\mathrm{card}(A) < \kappa$ and any $X \in \mathscr{S}$ there is some $Y \in \mathscr{S}$ such that $X \cup A \subseteq Y$ and Y/X is free.

If $\mathrm{gen}(M) = \mu$ is singular, Shelah's compactness theorem applies: M is free provided M is μ-free ([EM, IV, Theorem 3.5]).

Let κ be a regular uncountable cardinal. If $\mathrm{gen}(M) = \kappa$, then (in most cases) no such result holds and there is incompactness of various kinds. Of course, if R is a free ideal ring, then any strongly κ-free module is κ-free. For example, this is the case when $R = \mathbb{Z}$, the ring of all integers. On the other hand, for an arbitrary ring R, each κ^+-free module is strongly κ-free ([EM, IV, Theorem 3.4]).

It was an open problem ([EM, p. 87]) whether each strongly κ-free module is κ-free. First, we obtain a positive solution for $\kappa = \aleph_1$ and $\mathrm{gen}(M) \leq \aleph_1$,

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provided either R is countable or all projective modules are free. Our main result then gives a negative solution to the problem for each $\kappa > \aleph_1$.

More specifically, for each $\aleph_0 \leq \lambda \leq \kappa$, we say that

M is (κ, λ)*-free* provided there is a set, $\mathscr{C}$, of free submodules of M such that (1) and (2) from the definition of κ-free hold, and (3_λ) $\mathscr{C}$ is closed under unions of well-ordered chains of length $< \lambda$.

Clearly, (κ, λ)-free implies (κ, λ')-free for all $\aleph_0 \leq \lambda' \leq \lambda \leq \kappa$. Moreover, M is κ-free iff M is (κ, κ)-free. In Proposition 1, we show that any $\leq \kappa$ generated strongly κ-free module is $(\kappa, \aleph_1)$-free, provided either $\text{card}(R) < \kappa$ or all projective modules are free. On the other hand, for each $\kappa > \aleph_1$, we prove that there are a non-left-perfect ring R_κ and a non-projective left ideal I_κ such that $\text{gen}(I_\kappa) = \kappa$, I_κ is strongly κ-free and $(\kappa, \aleph_1)$-free, but not $(\kappa, \aleph_2)$-free (Theorem 8).

As an immediate consequence, we obtain our

Theorem. *For each regular cardinal $\kappa > \aleph_1$, there are a non-left-perfect ring R_κ and a non-projective strongly κ-free module M_κ with $\text{gen}(M_\kappa) = \kappa$ such that M_κ is not κ-free.*

Following [EM, p. 224], given a non-left-perfect ring R, we denote by $\text{Inc}'(R)$ the incompactness spectrum of R, i.e., the set of all uncountable cardinals λ such that there is a non-projective λ-free module M with $\text{gen}(M) \leq \lambda$. Similarly, $\text{Sinc}'(R)$ is the strong incompactness spectrum of R, i.e., the set of all uncountable cardinals λ such that there is a non-projective strongly λ-free module M with $\text{gen}(M) \leq \lambda$. The fundamental result of Shelah, Eklof, and Mekler ([EM, VII, Corollary 3.13]) says that

$$\text{Inc}'(\mathbb{Z}) = \text{Sinc}'(\mathbb{Z}) = \bigcap_{R \in \mathscr{N}} \text{Sinc}'(R) = \bigcap_{R \in \mathscr{N}} \text{Inc}'(R),$$

$\mathscr{N}$ being the class of all non-left-perfect rings. A recent result of Magidor and Shelah ([EM, p. 191]) shows that assuming the existence of some large cardinals, there is a model of $ZFC + GCH$ such that $\aleph_{\omega^2+1} \notin \text{Inc}'(\mathbb{Z})$, and a model with $\lambda \notin \text{Inc}'(\mathbb{Z})$ for all $\lambda \geq$ the first cardinal fixed point. Recall that the main open problem ([EM, p. 453, Problem 1]) is whether $\text{Inc}'(R) = \text{Inc}'(\mathbb{Z})$ for all $R \in \mathscr{N}$ (in ZFC). Though we do not solve this problem here, we have a related corollary to Theorem 8 and [EM, IV, Theorem 3.2]:

Corollary. *For each regular uncountable cardinal κ, there is a non-left-perfect ring R_κ such that $\kappa \in \text{Sinc}'(R_\kappa)$. Moreover,*

(i) *if κ is weakly compact, then $\kappa \in \text{Sinc}'(R_\kappa) \setminus \bigcup_{R \in \mathscr{N}} \text{Inc}'(R)$,*

(ii) *in the first and second models of Magidor and Shelah,*

$$\text{Sinc}'(R_{\aleph_{\omega^2+1}}) \neq \text{Sinc}'(\mathbb{Z}), \quad \textit{and} \quad \text{Sinc}'(R_\lambda) \neq \text{Sinc}'(\mathbb{Z})$$

for all regular $\lambda \geq$ the first cardinal fixed point, respectively.

A ring R is *non-left-perfect* provided there is a strongly decreasing countably infinite chain of principal right ideals of R. R is *von Neumann regular* provided each $x \in R$ has a pseudo-inverse $y \in R$ (i.e., $xyx = x$). A set $\{e_\alpha ; \alpha < \kappa\}$ is a set of *orthogonal idempotents* of R if $0 \neq e_\alpha = e_\alpha^2$ and $e_\alpha e_\beta = 0$ for all $\alpha, \beta < \kappa$ such that $\alpha \neq \beta$. For $r \in R$, define $\text{Ann}_R(r) = \{r' \in R ; r'r = 0\}$.

Homomorphisms in module categories are written as acting on the opposite side from the scalars. The category of all (unitary left R-) modules is denoted by R-Mod. A system of modules $(M_\alpha; \alpha < \kappa)$ is said to be a *smooth chain* provided $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \kappa$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \kappa$. A smooth chain with $M = \bigcup_{\alpha<\kappa} M_\alpha$ is a κ-*filtration* of M provided $\operatorname{gen}(M_\alpha) < \kappa$ for all $\alpha < \kappa$.

A closed and cofinal (= unbounded) subset of a regular uncountable cardinal λ is said to be a *cub*. A subset $E \subseteq \lambda$ is stationary in λ provided $E \cap C \neq \varnothing$ for each cub C.

For basic properties of these notions, the reader is referred to [AF], [EM], and [G].

We start with the case of $\lambda = \aleph_1$:

Proposition 1. *Let R be an arbitrary ring and κ a regular uncountable cardinal. Assume that either* $\operatorname{card}(R) < \kappa$ *or all projective modules are free. Then each $\leq \kappa$ generated strongly κ-free module is $(\kappa, \aleph_1)$-free. In particular, strongly $\aleph_1$-free implies $\aleph_1$-free provided* $\operatorname{gen}(M) \leq \aleph_1$ *and either R is countable or all projective modules are free.*

Proof. By [EM, IV, Lemma 1.10 and Proposition 1.11], there is a κ-filtration, $(M_\alpha; \alpha < \kappa)$, of M such that for all $\alpha < \beta < \kappa$, $M_{\alpha+1}$ and $M_{\beta+1}/M_{\alpha+1}$ are free. Denote by $\mathscr{C}$ the set of all M_α, $\alpha < \kappa$, such that either α is non-limit or α is a limit ordinal of cofinality ω. Then each element of $\mathscr{C}$ is a free module. Clearly, $\mathscr{C}$ satisfies the conditions (1), (2), and ($3_{\aleph_1}$) from the definition of $(\kappa, \aleph_1)$-free. □

Now, we fix our notation for the rest of the paper in the following Definitions 2 and 4.

Definition 2. Let κ be a regular uncountable cardinal and E a stationary subset of κ consisting of limit ordinals. Let K be a skew-field. Denote by L the right linear K-space of dimension κ, and let $S = \operatorname{End}(L_K)$, i.e., S is the ring of all linear transformations of L. Put $T = \{f \in S; \operatorname{rank}(f) < \kappa\}$. It is well known that T is the unique maximal two-sided ideal of S. Put $R = S/T$.

Lemma 3. *R is a simple von Neumann regular ring and R is not left perfect. Moreover, a module $M \in R$-*Mod *is projective iff M is free.*

Proof. Since S is von Neumann regular and T is maximal, R is a simple von Neumann regular ring. Since $\kappa \geq \aleph_0$, R contains an infinite set of orthogonal idempotents, whence R is neither left nor right perfect. If $0 \neq P$ is a projective module, then $P \simeq \oplus \sum_{\alpha<\lambda} Rg_\alpha$ for a cardinal $\lambda > 0$ and non-zero idempotents $g_\alpha \in R$, $\alpha < \lambda$ ([AF, p. 300]). For each $\alpha < \lambda$, there is an idempotent $h_\alpha \in S$ with $R(h_\alpha + T) = Rg_\alpha$. Put $H_\alpha = \operatorname{Ker}(h_\alpha)$ and $H'_\alpha = \operatorname{Im}(h_\alpha)$. Then $L = H_\alpha \oplus H'_\alpha$ in Mod-K. Since $g_\alpha \neq 0$, there is a K-isomorphism, x_α, of H'_α onto L. Extending x_α to L by zero values on H_α, we obtain $s_\alpha \in S$ such that $Ss_\alpha = Sh_\alpha$ and $\operatorname{Ann}_S(s_\alpha) = 0$. Moreover, $\operatorname{Ann}_R(s_\alpha + T) = T$, whence $Rg_\alpha = R(h_\alpha + T) = R(s_\alpha + T) \simeq R$. □

Definition 4. Let $B = \{b_{\beta\gamma}; (\beta, \gamma) \in \kappa \times \kappa\}$ be a right K-basis of L. For each $\alpha < \kappa$, define $c_\alpha, d_\alpha \in S$ by

$c_\alpha(b_{\beta\gamma}) = b_{\beta\gamma}$ provided $\beta \leq \alpha$ and $\gamma < \kappa$, and $c_\alpha(b_{\beta\gamma}) = 0$ otherwise,
and by

$d_\alpha(b_{\alpha\gamma}) = b_{\alpha\gamma}$ provided $\gamma < \kappa$, and $d_\alpha(b_{\beta\gamma}) = 0$ otherwise.

Put $e_\alpha = c_\alpha + T$ and $f_\alpha = d_\alpha + T$, $\alpha < \kappa$. Clearly, $\{f_\alpha ; \alpha < \kappa\}$ is a set of orthogonal idempotents of R, and e_α, $\alpha < \kappa$, are idempotents in R such that $Rf_0 = Re_0$, $Rf_\alpha \subseteq Re_\alpha \subset Re_\beta$, and $Re_\alpha \cap Rf_\beta = 0$ whenever $\alpha < \beta < \kappa$.

Define a system of left ideals $(I_\alpha ; \alpha < \kappa)$ as follows: $I_0 = 0$, $I_{\alpha+1} = Re_\alpha$ provided $\alpha \in E$, $I_{\alpha+1} = I_\alpha + Rf_\alpha$ provided $\alpha \notin E$, and $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$ provided α is a limit ordinal. Finally, put $I = \bigcup_{\alpha<\kappa} I_\alpha$.

Lemma 5. *I is a non-projective left ideal of R,* $\mathrm{gen}(I) = \kappa$, *and* $(I_\alpha ; \alpha < \kappa)$ *is a κ-filtration of I.*

Proof. Clearly, $I_{\alpha+1} \subseteq Re_\alpha$ for all $\alpha < \kappa$, and $(I_\alpha ; \alpha < \kappa)$ is a smooth chain of left ideals of R. Hence, $I_{\alpha+1} = I_\alpha \oplus Rf_\alpha$ for all $\alpha \notin E$, the chain is strictly increasing, and $\mathrm{gen}(I) = \kappa$. Moreover, $\mathrm{gen}(I_\alpha) \leq \mathrm{card}(\alpha) < \kappa$, whence $(I_\alpha ; \alpha < \kappa)$ is a κ-filtration of I.

For all $\alpha \in E$ and $\alpha < \beta < \kappa$, we have $I_\beta = Re_\alpha \oplus (I_\beta \cap R(1 - e_\alpha))$, and $I_\beta / I_\alpha \simeq I_{\alpha+1}/I_\alpha \oplus (I_\beta \cap R(1 - e_\alpha))$. Since I_α is not finitely generated, $I_{\alpha+1}/I_\alpha = Re_\alpha / I_\alpha$ is non-projective, and so is I_β / I_α.

Assume I is projective. By Lemma 3, there exist left ideals $C_\alpha \subset I$, $\alpha < \kappa$, such that $C_\alpha \simeq R$ for all $\alpha < \kappa$, and $I = \oplus \sum_{\alpha<\kappa} C_\alpha$. Put $J_\alpha = \oplus \sum_{\beta<\alpha} C_\beta$, for all $\alpha < \kappa$. Then $(J_\alpha ; \alpha < \kappa)$ is a κ-filtration of I, and $C = \{\alpha < \kappa ; I_\alpha = J_\alpha\}$ is a cub. Take $\alpha \in E \cap C$ and $\beta \in E \cap C \cap D$, where $D = \{\gamma < \kappa ; \alpha < \gamma\}$ is a cub, too. Then $I_{\alpha+1}/I_\alpha$ is a non-projective summand of $I_\beta / I_\alpha = J_\beta / J_\alpha \simeq \oplus \sum_{\alpha \leq \gamma < \beta} C_\gamma$, a contradiction. $\square$

As pointed out by the referee, the proof of the non-projectivity of I can also be accomplished in analogy to the proof of [EM, VII, Corollary 3.13]. Hence, the fact that all projective modules are free is not essential here.

Lemma 6. (i) *Let $\alpha = 0$ or $\alpha \in E$, and put $\alpha' = \min\{\beta \in E ; \alpha < \beta\}$. Then I_γ and I_δ / I_γ are free whenever $\alpha < \gamma < \delta < \alpha'$.*

(ii) *If γ is a limit ordinal and $\gamma < \delta < \kappa$ is such that there exists $\alpha \in E$ with $\gamma < \alpha < \delta$, then I_δ / I_γ is not projective.*

(iii) *Let $\alpha = 0$ or $\alpha \in E$, and let $0 < \nu < \aleph_1$. Then $I_{\alpha+\nu}$ is free.*

(iv) *If $\aleph_0 < \lambda < \kappa$, λ is a regular cardinal, and $E \cap \lambda$ is stationary in λ, then I_λ is not projective.*

Proof. (i) By induction on γ satisfying $\alpha + 1 \leq \gamma < \alpha'$, we get $I_\gamma = Re_\alpha \oplus \oplus \sum_{\alpha<\beta<\gamma} Rf_\beta$. Hence, $I_\delta / I_\gamma \simeq \oplus \sum_{\gamma \leq \beta < \delta} Rf_\beta$.

(ii) Since $I_{\alpha+1} = Re_\alpha$ is a summand of I_δ, and I_γ is not finitely generated, the module I_δ / I_γ has a non-projective summand isomorphic to $I_{\alpha+1}/I_\gamma$.

(iii) We prove the assertion by induction on ν. For $\nu = 1$, we have either $I_{\alpha+1} = Rf_0$ or $I_{\alpha+1} = Re_\alpha$. If $I_{\alpha+\nu}$ is free, then either $I_{\alpha+\nu+1} = I_{\alpha+\nu} \oplus Rf_{\alpha+\nu}$, or $I_{\alpha+\nu+1} = Re_{\alpha+\nu}$, and $I_{\alpha+\nu+1}$ is free, too. If $\nu < \aleph_1$ is a limit ordinal, then either (1) $\alpha + \nu = \sup_{n<\aleph_0}(\alpha_n)$, for some elements $\alpha_n \in E$, $n < \aleph_0$, or (2) $\sup_{\{\beta \in E, \beta < \alpha+\nu\}}(\beta) < \alpha + \nu$. In the case (1), we have $I_{\alpha+\nu} = \bigcup_{n<\aleph_0} Re_{\alpha_n}$. Then $I_{\alpha+\nu}$ is a countably generated left ideal of R, and $I_{\alpha+\nu}$ is free by Lemma 3 and [G, Corollary 2.15]. In the case (2), the assertion follows from the induction premise and part (i).

(iv) As in the proof of Lemma 5, I_δ / I_γ is non-projective for all $\gamma \in E \cap \lambda$ and $\gamma < \delta < \lambda$. Since $E \cap \lambda$ is stationary in λ, the same argument as in Lemma 5 shows that $I_\lambda = \bigcup_{\delta<\lambda} I_\delta$ is not projective. $\square$

Lemma 7. *I is strongly κ-free.*

Proof. Put $\mathscr{S} = \{0\} \cup \{I_{\alpha+1}\,;\, \alpha \in E\}$. We show that $\mathscr{S}$ is the system making I strongly κ-free. By Lemma 3 and Definition 4, $X \simeq R$ for any $X \in \mathscr{S} \setminus \{0\}$. Since $I = \bigcup_{\alpha \in E} I_{\alpha+1}$, for any $\alpha \in E$ and any set A with $\operatorname{card}(A) < \kappa$ there is some $\beta \in E$ with $I_{\alpha+1} \cup A \subseteq I_{\beta+1}$, and clearly $I_{\beta+1}/I_{\alpha+1}$ is free. □

Theorem 8. *I is a non-projective strongly κ-free left ideal of R with* $\operatorname{gen}(I) = \kappa$. *Moreover, I is $(\kappa, \aleph_1)$-free. If $\kappa > \aleph_1$, then I is not $(\kappa, \aleph_2)$-free.*

Proof. The first assertion follows from Lemmas 5 and 7. By Proposition 1 and Lemma 3, I is $(\kappa, \aleph_1)$-free.

Proving indirectly, assume $\kappa > \aleph_1$ and I is $(\kappa, \aleph_2)$-free. Let $\mathscr{C}$ be the system appearing in the definition of $(\kappa, \aleph_2)$-free. Take $J_0 \in \mathscr{C}$. By the premise, there is some $\alpha_0 \in E$ with $J_0 \subseteq Re_{\alpha_0}$. If $\beta < \aleph_1$ is limit, we take $J_\beta = \bigcup_{\gamma<\beta} J_\gamma$ and $\alpha_\beta \in E$ such that $J_\beta \subseteq Re_{\alpha_\beta}$. If $\beta < \aleph_1$ and $J_0 \subseteq Re_{\alpha_0} \subseteq \cdots \subseteq J_\beta \subseteq Re_{\alpha_\beta}$, we take $J_{\beta+1} \in \mathscr{C}$ and $\alpha_{\beta+1} \in E$ such that $Re_{\alpha_\beta} \subset J_{\beta+1} \subseteq Re_{\alpha_{\beta+1}}$. Put $J = \bigcup_{\beta<\aleph_1} J_\beta = \bigcup_{\beta<\aleph_1} Re_{\alpha_\beta}$. By the construction, $\operatorname{gen}(J) = \aleph_1$. By $(3_{\aleph_2})$, $J \in \mathscr{C}$, and there exist submodules C_γ, $\gamma < \aleph_1$, of J such that $C_\gamma \simeq R$, $\gamma < \aleph_1$, and $J = \oplus\sum_{\gamma<\aleph_1} C_\gamma$. Then there is some $\beta < \aleph_1$ such that $\oplus\sum_{\gamma<\aleph_0} C_\gamma \subset Re_{\alpha_\beta} \subset \oplus\sum_{\gamma<\aleph_1} C_\gamma$. Since $\oplus\sum_{\gamma<\aleph_1} C_\gamma = Re_{\alpha_\beta} \oplus X$, where $X = (\oplus\sum_{\gamma<\aleph_1} C_\gamma) \cap R(1-e_{\alpha_\beta})$, we have $\oplus\sum_{\aleph_0\le\gamma<\aleph_1} C_\gamma \simeq (Re_{\alpha_\beta}/\oplus\sum_{\gamma<\aleph_0} C_\gamma) \oplus X$, the first summand being non-projective, a contradiction. □

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