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A GAP THEOREM FOR ENDS OF COMPLETE MANIFOLDS

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ABSTRACT. Let (M^n, o) be a pointed open complete manifold with Ricci curvature bounded from below by $-(n-1)\Lambda^2$ (for $\Lambda \geq 0$) and nonnegative outside the ball $B(o, a)$. It has recently been shown that there is an upper bound for the number of ends of such a manifold which depends only on Λa and the dimension n of the manifold M^n. We will give a gap theorem in this paper which shows that there exists an $\varepsilon = \varepsilon(n) > 0$ such that M^n has at most two ends if $\Lambda a \leq \varepsilon(n)$. We also give examples to show that, in dimension $n \geq 4$, such manifolds in general do not carry any complete metric with nonnegative Ricci Curvature for any $\Lambda a > 0$.

1. INTRODUCTION

The Cheeger-Gromoll splitting theorem states that in a complete manifold of nonnegative Ricci curvature, a line splits off isometrically, i.e., any nonnegatively Ricci curved M^n is isometric to a Riemannian product $N^k \times \mathbf{R}^{n-k}$, where N does not contain a line (cf. [CG]). In particular, such a manifold has at most two ends. Recently, the first-named author and independently Li and Tam have shown that a complete manifold with nonnegative Ricci curvature outside a compact set has at most finitely many ends [C, LT]. At about the same time, Liu has also given a proof of the same theorem with an additional condition that there is a lower bound on sectional curvature [L], which was removed shortly after the appearance of [C]. In this paper, we consider manifolds with nonnegative Ricci curvature outside a compact set and prove the following gap theorem.

Theorem. *Given $n > 0$, there exists an $\varepsilon = \varepsilon(n) > 0$ such that for all pointed open complete manifolds (M^n, o) with Ricci curvature bounded from below by $-(n-1)\Lambda^2$ (for $\Lambda \geq 0$) and nonnegative outside the ball $B(o, a)$, if $\Lambda a \leq \varepsilon(n)$, then M^n has at most two ends.*

A natural question one would like to ask is whether this theorem can be improved so that M^n must carry a complete metric with nonnegative Ricci curvature. Indeed, it is easy to see by volume comparison that the answer to the above question is affirmative in dimension 2 since the Euler number of such

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a 2-dimensional complete manifold is an upper bound for the total curvature integral. However, such a gap theorem is the best one can have in dimensions higher than 3 as illustrated by the following examples.

For any $\varepsilon > 0$, by gluing two sharp cones together at the singular point, it is easy to construct a complete metric on $R \times S^{n-2}$, $n \geq 4$, with Ricci curvature bounded from below by $-\varepsilon$ and with nonnegative sectional curvature away from a metric ball of radius 1. By applying the metric surgery techniques as in [SY] to the manifold $S^1 \times R \times S^{n-2}$, one obtains an n-dimensional complete manifold M of infinite homotopy type with exactly two ends and with Ricci curvature bounded from below by $-\varepsilon$ and with nonnegative Ricci curvature outside a metric ball of radius 1. M certainly cannot carry any complete metric with nonnegative Ricci curvature since the Cheeger-Gromoll splitting theorem implies that a nonnegatively Ricci curved manifold with exactly two ends must split isometrically into the product of R with a closed manifold and therefore has finite homotopy type.

The above examples are not valid in dimension 3 since the kind of metric surgery lemmas are not available. Therefore, the following problem is of particular interest:

Does there exist an $\varepsilon > 0$ such that if (M, o) is a pointed noncompact complete 3-dimensional manifold with Ricci curvature bounded from below by $-\varepsilon$ and nonnegative outside the unit metric ball $B(o, 1)$, then M carries a complete metric with nonnegative Ricci curvature?

2. Proof of the theorem

There are various (but equivalent) definitions of an end of a manifold. For the sake of our argument, we use the following (compare with [A]).

Definition 2.1. Two rays γ_1 and γ_2 starting at the base point o are called cofinal, if for any $r \geq 0$ and all $t \geq r$, $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B(o, r)$. An equivalence class of cofinal rays is called an end of M. We will denote by $[\gamma]$ the equivalence class of γ.

Notice that the above definition does not depend on the base point o and the particular complete metric on M. Thus the number of ends of M is a topological invariant of M.

The following lemma is a refined version of Proposition 2.2 in [C] and can be proved by the same argument.

Lemma 2.2. *Let M be as in the theorem. If $[\gamma_1]$ and $[\gamma_2]$ are two different ends of M, then for any $t_1, t_2 \geq 0$, $d(\gamma_1(t_1), \gamma_2(t_2)) \geq t_1 + t_2 - 2a$.*

In what follows, let M^n be as in the theorem. By scaling, we may assume that $\mathrm{Ric}(M^n) \geq -(n-1)$.

Following Abresch and Gromoll in [AG], let $\phi(x)$ be the function defined on $B_{-1}(o, 1) - \{o\}$, the truncated unit ball in the hyperbolic space $\mathbf{H}^n$, with the following property:

$$\Delta\phi = 2(n-1),$$
$$\phi|_{\partial B_{-1}(1)} = 0.$$

It is easy to see that $\phi(x) = G(d(o, x))$, where

$$G(r) = 2(n-1)\int_r^1 \int_t^1 \left(\frac{\sinh s}{\sinh t}\right)^{n-1} ds\, dt.$$

Given a continuous function u: $M \to R$ and $x \in M$, a continuous function u_x: $M \to R$ is called an upper barrier of u at x if $u_x(x) = u(x)$ and $u \le u_x$. The following lemma is a slight generalization of Theorem 2.1 in [AG].

Lemma 2.3. *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below by $-(n-1)$. Then there exist an $\varepsilon = \varepsilon(n) > 0$ and a $\delta = \delta(n) > 0$ such that*

$$u(x) < 2 - 2\delta - 4\varepsilon$$

for all $x \in S(o, 1-\delta)$ if u: $M^n \to \mathbf{R}$ is a continuous function which satisfies the following properties:

$$(1) \qquad u(o) = 0,$$

$$(2) \qquad u \ge -2\varepsilon,$$

$$(3) \qquad \mathrm{dil}(u) \le 2,$$

$$(4) \qquad \Delta u \le 2(n-1),$$

where $\mathrm{dil}(u) = \sup_{x \ne y} |u(x) - u(y)|/d(x, y)$ *and the last inequality is in the barrier sense, that is, for any $x \in M$ and $\alpha > 0$, there is an upper barrier of u at x, $u_{x,\alpha}$, such that $u_{x,\alpha}$ is smooth near x and $\Delta u_{x,\alpha}(x) \le 2(n-1) + \alpha$.*

Proof. Consider $H(r) = 2r + G(r)$. Notice that $G(1) = 0$ and $G'(1) = 0$. Hence $H(1) = 2$ and $H'(r) > 0$ for r close to 1, and therefore there exists a c such that $0 < c < 1$ and $H(c) < 2$. Now choose $\delta = \delta(n)$ and $\varepsilon = \varepsilon(n)$ such that

$$(5) \qquad 0 < \delta < \tfrac{1}{2}\min\{2 - H(c), 1 - c\}$$

and

$$(6) \qquad 0 < \varepsilon < \tfrac{1}{2}\min\{G(1-\delta), 2 - H(c) - 2\delta\}.$$

Consider the function $v(y) = u(y) - G(d(x, y))$ on the annulus $B(x, 1)\backslash B(x, c)$. The well-known Laplacian comparison theorem for distance functions (cf. [EH]) implies that $\Delta v \le 0$ (in the barrier sense). By the maximum principle [EH], v achieves its minimum on the boundary of the annulus. Since o is an interior point of the domain by (5) and $v(o) = u(o) - G(d(o, x)) = -G(1-\delta) < -2\varepsilon$ by (6), there exists a point z on the boundary of the domain such that $v(z) < -2\varepsilon$. But on $S(x, 1)$, $v = u - G(1) = u \ge -2\varepsilon$ by (2). Hence $z \in S(x, c)$. Combining this with (3) and (6), we conclude that

$$u(x) \le u(z) + 2c = v(z) + H(c) < 2 - 2\delta - 4\varepsilon.$$

This proves Lemma 2.3.

Remark 2.4. For a ray γ in M, let b_γ be the associated Busemann function, i.e.,

$$b_\gamma(x) = \lim_{t\to\infty}(d(\gamma(t), x) - t).$$

It is well known (e.g., see [EH]) that, in the barrier sense, $\Delta b_\gamma \le n - 1$. We are now in position to prove the theorem.

Proof of the theorem. Let M^n be as in the theorem with $\Lambda = 1$. Let $\varepsilon = \varepsilon(n)$ be as in Lemma 2.3. We need to show that when $a \leq \varepsilon$, M^n has at most two ends. Suppose not. Let $[\gamma_1]$, $[\gamma_2]$, and $[\gamma_3]$ be three different ends. Consider $u := b_{\gamma_1} + b_{\gamma_2}$. We claim that u satisfies the conditions in Lemma 2.3. As a matter of fact, (1) and (3) are clear, (4) is by Remark 2.4, and (2) is a consequence of the triangle inequality and Lemma 2.2. From Lemma 2.3, we conclude that

$$u(\gamma_3(1-\delta)) < 2 - 2\delta - 4\varepsilon. \tag{7}$$

On the other hand, it follows from Lemma 2.2 that for any $t \geq 0$,

$$u(\gamma_3(t)) \geq 2t - 4a.$$

In particular,

$$u(\gamma_3(1-\delta)) \geq 2(1-\delta) - 4a \geq 2 - 2\delta - 4\varepsilon.$$

This clearly contradicts (7) and hence completes the proof of the theorem.

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