A GAP THEOREM FOR ENDS OF COMPLETE MANIFOLDS

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Abstract. Let \((M^n, \sigma)\) be a pointed open complete manifold with Ricci curvature bounded from below by \(-(n - 1)\Lambda^2\) (for \(\Lambda \geq 0\)) and nonnegative outside the ball \(B(\sigma, a)\). It has recently been shown that there is an upper bound for the number of ends of such a manifold which depends only on \(\Lambda a\) and the dimension \(n\) of the manifold \(M^n\). We will give a gap theorem in this paper which shows that there exists an \(\epsilon = \epsilon(n) > 0\) such that \(M^n\) has at most two ends if \(\Lambda a < \epsilon(n)\). We also give examples to show that, in dimension \(n \geq 4\), such manifolds in general do not carry any complete metric with nonnegative Ricci Curvature for any \(\Lambda a > 0\).

1. Introduction

The Cheeger-Gromoll splitting theorem states that in a complete manifold of nonnegative Ricci curvature, a line splits off isometrically, i.e., any nonnegatively Ricci curved \(M^n\) is isometric to a Riemannian product \(N^k \times \mathbb{R}^{n-k}\), where \(N\) does not contain a line (cf. [CG]). In particular, such a manifold has at most two ends. Recently, the first-named author and independently Li and Tam have shown that a complete manifold with nonnegative Ricci curvature outside a compact set has at most finitely many ends [C, LT]. At about the same time, Liu has also given a proof of the same theorem with an additional condition that there is a lower bound on sectional curvature [L], which was removed shortly after the appearance of [C]. In this paper, we consider manifolds with nonnegative Ricci curvature outside a compact set and prove the following gap theorem.

Theorem. Given \(n > 0\), there exists an \(\epsilon = \epsilon(n) > 0\) such that for all pointed open complete manifolds \((M^n, \sigma)\) with Ricci curvature bounded from below by \(-(n - 1)\Lambda^2\) (for \(\Lambda \geq 0\)) and nonnegative outside the ball \(B(\sigma, a)\), if \(\Lambda a \leq \epsilon(n)\), then \(M^n\) has at most two ends.

A natural question one would like to ask is whether this theorem can be improved so that \(M^n\) must carry a complete metric with nonnegative Ricci curvature. Indeed, it is easy to see by volume comparison that the answer to the above question is affirmative in dimension 2 since the Euler number of such
a 2-dimensional complete manifold is an upper bound for the total curvature integral. However, such a gap theorem is the best one can have in dimensions higher than 3 as illustrated by the following examples.

For any \( \varepsilon > 0 \), by gluing two sharp cones together at the singular point, it is easy to construct a complete metric on \( R \times S^{n-2}, \ n \geq 4 \), with Ricci curvature bounded from below by \( -\varepsilon \) and with nonnegative sectional curvature away from a metric ball of radius 1. By applying the metric surgery techniques as in [SY] to the manifold \( S^1 \times R \times S^{n-2} \), one obtains an \( n \)-dimensional complete manifold \( M \) of infinite homotopy type with exactly two ends and with Ricci curvature bounded from below by \( -\varepsilon \) and with nonnegative Ricci curvature outside a metric ball of radius 1. \( M \) certainly cannot carry any complete metric with nonnegative Ricci curvature since the Cheeger-Gromoll splitting theorem implies that a nonnegatively Ricci curved manifold with exactly two ends must split isometrically into the product of \( R \) with a closed manifold and therefore has finite homotopy type.

The above examples are not valid in dimension 3 since the kind of metric surgery lemmas are not available. Therefore, the following problem is of particular interest:

Does there exist an \( \varepsilon > 0 \) such that if \((M, o)\) is a pointed noncompact complete 3-dimensional manifold with Ricci curvature bounded from below by \( -\varepsilon \) and nonnegative outside the unit metric ball \( B(o, 1) \), then \( M \) carries a complete metric with nonnegative Ricci curvature?

2. Proof of the theorem

There are various (but equivalent) definitions of an end of a manifold. For the sake of our argument, we use the following (compare with [A]).

**Definition 2.1.** Two rays \( \gamma_1 \) and \( \gamma_2 \) starting at the base point \( o \) are called cofinal, if for any \( r \geq 0 \) and all \( t \geq r \), \( \gamma_1(t) \) and \( \gamma_2(t) \) lie in the same component of \( M - B(o, r) \). An equivalence class of cofinal rays is called an end of \( M \). We will denote by \([\gamma]\) the equivalence class of \( \gamma \).

Notice that the above definition does not depend on the base point \( o \) and the particular complete metric on \( M \). Thus the number of ends of \( M \) is a topological invariant of \( M \).

The following lemma is a refined version of Proposition 2.2 in [C] and can be proved by the same argument.

**Lemma 2.2.** Let \( M \) be as in the theorem. If \([\gamma_1]\) and \([\gamma_2]\) are two different ends of \( M \), then for any \( t_1, t_2 \geq 0 \), \( d(\gamma_1(t_1), \gamma_2(t_2)) \geq t_1 + t_2 - 2a \).

In what follows, let \( M^n \) be as in the theorem. By scaling, we may assume that \( \text{Ric}(M^n) \geq -(n - 1) \).

Following Abresch and Gromoll in [AG], let \( \phi(x) \) be the function defined on \( B_{-1}(o, 1) - \{o\} \), the truncated unit ball in the hyperbolic space \( \mathbb{H}^n \), with the following property:

\[
\Delta \phi = 2(n - 1), \\
\phi \big|_{\partial B_{-1}(1)} = 0.
\]
It is easy to see that \( \phi(x) = G(d(o, x)) \), where
\[
G(r) = 2(n-1) \int_r^1 \int_t^1 \left( \frac{\sinh s}{\sinh t} \right)^{n-1} ds \, dt.
\]

Given a continuous function \( u: M \to R \) and \( x \in M \), a continuous function \( u_x: M \to R \) is called an upper barrier of \( u \) at \( x \) if \( u_x(x) = u(x) \) and \( u \leq u_x \). The following lemma is a slight generalization of Theorem 2.1 in [AG].

**Lemma 2.3.** Let \( M^n \) be a complete Riemannian manifold with Ricci curvature bounded from below by \(-(n - 1)\). Then there exist an \( \varepsilon = \varepsilon(n) > 0 \) and a \( \delta = \delta(n) > 0 \) such that
\[
u(x) < 2 - 2\delta - 4\varepsilon
\]
for all \( x \in S(o, 1 - \delta) \) if \( u: M^n \to R \) is a continuous function which satisfies the following properties:

1. \( u(o) = 0 \),
2. \( u \geq -2\varepsilon \),
3. \( \text{dil}(u) \leq 2 \),
4. \( \Delta u \leq 2(n-1) \),

where \( \text{dil}(u) = \sup_{x \neq y} |u(x) - u(y)|/d(x, y) \) and the last inequality is in the barrier sense, that is, for any \( x \in M \) and \( \alpha > 0 \), there is an upper barrier of \( u \) at \( x \), \( u_{x, \alpha} \), such that \( u_{x, \alpha} \) is smooth near \( x \) and \( \Delta u_{x, \alpha}(x) \leq 2(n-1) + \alpha \).

**Proof.** Consider \( H(r) = 2r + G(r) \). Notice that \( G(1) = 0 \) and \( G'(1) = 0 \). Hence \( H(1) = 2 \) and \( H'(r) > 0 \) for \( r \) close to 1, and therefore there exists a \( c \) such that \( 0 < c < 1 \) and \( H(c) < 2 \). Now choose \( \delta = \delta(n) \) and \( \varepsilon = \varepsilon(n) \) such that
\[
0 < \delta = \frac{1}{2} \min\{2 - H(c), 1 - c\}
\]
and
\[
0 < \varepsilon = \frac{1}{2} \min\{2 - H(c), 2 - H(c) - 2\delta\}.
\]

Consider the function \( v(y) = u(y) - G(d(x, y)) \) on the annulus \( B(x, 1) \setminus B(x, c) \). The well-known Laplacian comparison theorem for distance functions (cf. [EH]) implies that \( \Delta v \leq 0 \) (in the barrier sense). By the maximum principle [EH], \( v \) achieves its minimum on the boundary of the annulus. Since \( o \) is an interior point of the domain by (5) and \( v(o) = u(o) - G(d(o, x)) = -G(1 - \delta) < -2\varepsilon \) by (6), there exists a point \( z \) on the boundary of the domain such that \( v(z) < -2\varepsilon \). But on \( S(x, 1) \), \( v = u - G(1) = u \geq -2\varepsilon \) by (2). Hence \( z \in S(x, c) \). Combining this with (3) and (6), we conclude that
\[
u(x) \leq u(z) + 2c = v(z) + H(c) < 2 - 2\delta - 4\varepsilon.
\]

This proves Lemma 2.3.

**Remark 2.4.** For a ray \( \gamma \) in \( M \), let \( b_\gamma \) be the associated Busemann function, i.e.,
\[
b_\gamma(x) = \lim_{t \to \infty} (d(\gamma(t), x) - t).
\]

It is well known (e.g., see [EH]) that, in the barrier sense, \( \Delta b_\gamma \leq n - 1 \). We are now in position to prove the theorem.
Proof of the theorem. Let $M^n$ be as in the theorem with $\Lambda = 1$. Let $\varepsilon = \varepsilon(n)$ be as in Lemma 2.3. We need to show that when $a \leq \varepsilon$, $M^n$ has at most two ends. Suppose not. Let $[\gamma_1]$, $[\gamma_2]$, and $[\gamma_3]$ be three different ends. Consider $u := b_{\gamma_1} + b_{\gamma_2}$. We claim that $u$ satisfies the conditions in Lemma 2.3. As a matter of fact, (1) and (3) are clear, (4) is by Remark 2.4, and (2) is a consequence of the triangle inequality and Lemma 2.2. From Lemma 2.3, we conclude that

$$u(\gamma_3(1 - \delta)) < 2 - 2\delta - 4\varepsilon. \tag{7}$$

On the other hand, it follows from Lemma 2.2 that for any $t \geq 0$,

$$u(\gamma_3(t)) \geq 2t - 4a.$$

In particular,

$$u(\gamma_3(1 - \delta)) \geq 2(1 - \delta) - 4a \geq 2 - 2\delta - 4\varepsilon.$$

This clearly contradicts (7) and hence completes the proof of the theorem.

References


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