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VOLUME DENSITIES WITH THE MEAN VALUE PROPERTY FOR HARMONIC FUNCTIONS

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ABSTRACT. On a bounded domain U in $\mathbb{R}^d$ containing the origin, probability measures μ which have a density w with respect to Lebesgue measure and satisfy $h(0) = \int h\,d\mu$ for every bounded harmonic function on U are studied. A domain U is constructed such that $\inf w(U) = 0$ for any such measure. (This solves a problem proposed by A. Cornea.) If, however, U has smooth boundary, then μ having a density $w \in \mathscr{C}^\infty(U)$ which is bounded away from zero on U can be constructed. On the other hand, for arbitrary U it is always possible to choose a strictly positive $w \in \mathscr{C}^\infty(U)$ tending to zero at ∂U.

1. INTRODUCTION

Let U be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, and $0 \in U$. There are many positive measures μ on U such that $\mu(U) > 0$ and

$$(*) \qquad h(0) = \frac{1}{\mu(U)} \int_U h\,d\mu$$

holds for every $h \in \mathscr{H}_b(U)$, the set of bounded harmonic functions on U. For various purposes, such measures were investigated, e.g., in [CD, Fl, FL, Ga, Hi, Sm, Za].

If desired, μ can be chosen to be absolutely continuous with respect to Lebesgue measure λ on U, say, $\mu = w\lambda$. (Indeed, take, e.g., $w = 1/\lambda(B)$ on a ball B contained in U and $w = 0$ elsewhere in U.)

In a discussion during the International Conference on Potential Theory (Nagoya, 1990), A. Cornea raised the problem whether there always exists a function w fulfilling $(*)$ that is *bounded away from zero* on U.

A counterexample constructed in §2 of this paper provides a negative solution to this problem. Our domain U is regular for the Dirichlet problem, and, actually, its boundary behaves badly near a single point only. It turns out that the problem is related to the integrability of positive harmonic functions on U with respect to λ.

Section 3 contains positive results: There always exists a *strictly positive* $w \in \mathscr{C}^\infty(U)$ satisfying $(*)$. If, moreover, the boundary of U is smooth enough

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(e.g., of class $\mathscr{C}^{1+\varepsilon}$), then a function $w \in \mathscr{C}^\infty(U)$ that is bounded away from zero can be produced. The latter result is based on [Wi].

2. A COUNTEREXAMPLE

For $z \in \mathbb{R}^d$ and $r > 0$ denote $B_r(z) = \{z' \in \mathbb{R}^d : \|z' - z\| < r\}$. For $n \in \mathbb{N} \cup \{0\}$ let $y_n = (2^{-n}, 0, \dots, 0)$, $x_n = (y_n + y_{n+1})/2$, $r_n = 2^{-n-2}$,

$$s_n = \begin{cases} \exp(-4^{2n+1}) & \text{if } d = 2, \\ 4^{-d/(d-2)(n+1)} & \text{if } d > 2, \end{cases}$$

$V_n = B_{r_n}(x_n)$, $W_n = B_{s_n}(y_n)$, and

$$U = \bigcup_{n=0}^{\infty} V_n \cup \bigcup_{n=1}^{\infty} W_n .$$

Notice that U is a regular domain contained in $B_1(0)$.

Denote $H(U) = \{h \in \mathscr{C}(\overline{U}) : h|_U \text{ harmonic}\}$, and write $H^+(U)$ for the set of positive functions from $H(U)$. As usual, for a resolutive numerical function f on ∂U, let $H_U f$ denote the corresponding solution of the Dirichlet problem.

2.1. **Proposition.** $\sup\{\int_U h\, d\lambda : h \in H^+(U),\ h(x_0) \le 1\} = \infty$.

Proof. Let $B = B_1(0)$, $a_+ = (1, 0, \dots, 0)$, $a_- = (-1, 0, \dots, 0)$,

$$D = B \setminus (B_{1/2}(a_+) \cup B_{1/2}(a_-)), \qquad L = \partial B \setminus (B_{1/4}(a_+) \cup B_{1/4}(a_-)),$$

and $g = H_B 1_L$. Denote $\alpha = \inf g(D)$, $\beta = \lambda(D)$. It is easy to see that $\alpha > 0$.

In the following considerations sweeping and balayage are understood with respect to (the $\mathscr{P}$-harmonic space) $X = B_2(0)$ if $d = 2$ and with respect to $X = \mathbb{R}^d$ if $d > 2$.

Let $x = x_0$, and fix $n \in \mathbb{N}$. If $d = 2$, then

$$R_1^{W_n}(x) \le \frac{1 - \ln \|y_n - x\|}{1 - \ln s_n} < 4 \left(\ln \frac{1}{s_n} \right)^{-1} \le 4^{-2n}.$$

For $d > 2$ we have

$$R_1^{W_n}(x) = \frac{s_n^{d-2}}{\|x - y_n\|^{d-2}} \le 4^{-d(n+1)} 4^{d-2} < 4^{-dn}.$$

The measure $\mu_x := \varepsilon_x^{\complement(U \setminus \overline{V}_n)} = \varepsilon_x^{\complement U \cup \overline{V}_n}$ is supported by the boundary of the connected component of $U \setminus \overline{V}_n$ containing x. Defining $A_n := \overline{V}_n \cap \overline{W}_n$ we hence know that $\mu_x(\overline{V}_n) = \mu_x(A_n)$. General properties of balayage (see [BH, p. 317]) imply that $\varepsilon_x^{\complement U}(\overline{V}_n) \le \mu_x(\overline{V}_n)$ and $\mu_x(A_n) \le \varepsilon_x^{A_n}(X)$ (of course, one could just as well use the boundary minimum principle to obtain these inequalities). Since $\varepsilon_x^{A_n}(X) \le R_1^{A_n}(x) \le R_1^{W_n}(x) \le 4^{-dn}$, we conclude that $\varepsilon_x^{\complement U}(\overline{V}_n) < 4^{-dn}$. Choose $f_n \in \mathscr{C}(\partial U)$ such that $0 \le f_n \le 4^{dn}$, $f_n = 0$ on $\complement \overline{V}_n$, and $f_n = 4^{dn}$ on $L_n = \partial V_n \setminus (B_{r_n/4}(y_n) \cup B_{r_n/4}(y_{n+1}))$. Denoting

$$D_n = V_n \setminus (B_{r_n/2}(y_n) \cup B_{r_n/2}(y_{n+1})), \qquad g_n = H_{V_n} 1_{L_n}$$

we get (using the invariance properties of harmonic functions)

$$\int_{D_n} g_n\, d\lambda \ge \alpha \beta r_n^d .$$

Since $H_U f_n \geq 4^{dn} g_n$ on V_n by the minimum principle, we obtain that

$$\int_U H_U f_n \, d\lambda \geq 4^{dn} \int_{D_n} g_n \, d\lambda \geq \alpha\beta 2^{d(n-2)}.$$

On the other hand, for every $n \in \mathbb{N}$,

$$H_U f_n(x) \leq 4^{dn} \varepsilon_x^{\complement U}(\overline{V}_n) \leq 1,$$

finishing the proof. □

Obviously, (2.1) implies the existence of a harmonic function $h \geq 0$ on U which is not integrable (it suffices to choose $h_n \in H^+(U)$ such that $h_n(x_0) = 1$, $\int_U h_n \, d\lambda \geq 2^n$, and to take $h = \sum_{n=1}^{\infty} 2^{-n} h_n$). In fact, our proof yields a stronger result:

2.2. **Proposition.** *There exists* $f\colon \partial U \to [0, \infty]$ *continuous (and finite) on* $\partial U \setminus \{0\}$ *such that* $H_U f(x_0) \leq 1$ *and* $\int_U H_U f \, d\lambda = \infty$.

Proof. Define f_n as above and take $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. □

Moreover, it is an immediate consequence of (2.1) that we have constructed a counterexample to the question of A. Cornea:

2.3. **Corollary.** *Let* ν *be a finite positive measure on* U *and* $a \geq 0$, *and assume that* $\int_U h \, d(a\lambda + \nu) = h(x_0)$ *for all* $h \in H(U)$. *Then* $a = 0$.

Proof. Let $h \in H^+(U)$, $h(x_0) \leq 1$. By assumption,

$$a \int_U h \, d\lambda \leq \int_U h \, d(a\lambda + \nu) \leq 1,$$

hence $a = 0$ by (2.1). □

3. Strictly positive densities

3.1. **Lemma.** *Let* $U \subset \mathbb{R}^d$ *be a bounded domain,* $0 \in U$. *Then there exists a family* $(U_\alpha)_{\alpha>0}$ *of regular subdomains of* U *such that* $\overline{U}_\beta \subset U_\alpha$ *for all* $0 < \alpha < \beta < \infty$, $\bigcap_{\alpha>0} U_\alpha = \{0\}$, $\bigcup_{\alpha>0} \partial U_\alpha = U \setminus \{0\}$.

Proof. If U is regular, it suffices to consider $g = G_U(0, \cdot)$ (G_U being the Green function for U) and to define $U_\alpha = \{g > \alpha\}$. Indeed, since $\lim_{x\to z} g(x) = 0$ for every $z \in \partial U$, we know that $\partial U_\alpha \subset U$, $g = \alpha$ on ∂U_α. In particular, U_α is regular, since $g - \alpha$ is a barrier for the points of ∂U_α. If W were a connected component of U_α not containing 0 then $g = \alpha$ on ∂W, hence $g = \alpha$ on W by the minimum principle, thus $W \cap U_\alpha = \varnothing$, a contradiction. So U_α is connected.

In the general case we take an open ball $W_0 = B_r(0)$ such that $\overline{W}_0 \subset U$ and define

$$U_\alpha = B_{r/\alpha}(0), \qquad \alpha > 1.$$

Moreover, we choose an increasing sequence (W_j) of domains in U such that $\overline{W}_j \subset W_{j+1}$ and $W_{j+1} \setminus \overline{W}_j$ is regular for all $j \geq 0$, $\bigcup_{j=0}^{\infty} W_j = U$, and define

$$U_t = \begin{cases} W_j, & t = 2^{-j}, \ j = 0, 1, 2, \ldots, \\ \overline{W}_j \cup \{H_{W_{j+1}\setminus\overline{W}_j} 1_{\partial W_j} > 2^{j+1}t - 1\}, & 2^{-(j+1)} < t < 2^{-j}, \ j = 0, 1, 2, \ldots. \end{cases}$$

Arguing similarly as before it is easily verified that the family $(U_\alpha)_{\alpha>0}$ has the desired properties. (The sequence (W_j) can be obtained in the following way: Take an increasing sequence (K_n) of connected compact sets such that $\bigcup_{n=1}^\infty K_n = U$. Suppose that $j \in \mathbb{N}$ and a domain W_{j-1} satisfying $0 \in \overline{W}_{j-1} \subset U$ has already been chosen. Take $k \in \mathbb{N}$ such that $2^{-k}\sqrt{d} <$ dist$(K_j \cup \overline{W}_{j-1}, \complement U)$, and let W_j be the interior of the union of all cubes of the form $\prod_{j=1}^d [m_j 2^{-k}, (m_j+1)2^{-k}[$, $m_j \in \mathbb{Z}$, intersecting $K_j \cup \overline{W}_{j-1}$. Then $\overline{W}_j \subset U$ by the choice of k and W_j is a connected neighborhood of the (connected) set $K_j \cup \overline{W}_{j-1}$. The regularity of the sets $W_{j+1} \setminus \overline{W}_j$ follows from the fact that open cubes Q are not thin at points $z \in \partial Q$.) □

3.2. **Theorem.** *Let $U \subset \mathbb{R}^d$ be a bounded domain, $0 \in U$. Then there exists a strictly positive function $w \in \mathscr{C}^\infty(U)$ tending to 0 at ∂U such that $\int_U hw\,d\lambda = h(0)$ for all $h \in \mathscr{H}_b(U)$.*

Proof. Choose a family $(U_\alpha)_{\alpha>0}$ according to (3.1), and fix a radially symmetric $\mathscr{C}^\infty$-function $\tau \ge 0$ on $\mathbb{R}^d$ such that $\{\tau > 0\} = B_1(0)$ and $\int_{\mathbb{R}^d} \tau\,d\lambda = 1$. For $y \in U$ put $\rho(y) = \frac{1}{2}\operatorname{dist}(y, \complement U)$ and define

$$\tau_y : x \mapsto (1/\rho(y))^d \tau((x-y)/\rho(y)), \qquad x \in \mathbb{R}^d.$$

Of course, τ_y is a $\mathscr{C}^\infty$-function on $\mathbb{R}^d$, $\{\tau > 0\} = B_{\rho(y)}(y)$, and $\int_{\mathbb{R}^d} \tau_y\,d\lambda = 1$. For every $\alpha > 0$ define $\mu_\alpha = \varepsilon_0^{\complement U_\alpha}$ and

$$w_\alpha : x \mapsto \int_{\partial U_\alpha} \tau_y(x)\,d\mu_\alpha(y), \qquad x \in \mathbb{R}^d.$$

Then $w_\alpha \in \mathscr{C}^\infty(U)$ and $\{w_\alpha > 0\} = \bigcup\{B_{\rho(y)}(y) : y \in \partial U_\alpha\}$ since ∂U_α is the support of μ_α. Moreover, $\int_U hw_\alpha\,d\lambda = h(0)$ for every $h \in \mathscr{H}_b(U)$ since $\int h\,d\mu_\alpha = h(0)$ and τ is radially symmetric.

There exists a decreasing sequence (α_n) of strictly positive real numbers such that $\lim_{n\to\infty} \alpha_n = 0$ and

$$B_{\rho(0)}(0) \cup \bigcup_{n=1}^\infty \{w_{\alpha_n} > 0\} = U.$$

Define

$$v = \tau_0 + \sum_{n=1}^\infty \frac{1}{2^n \sup w_{\alpha_n}(U)} w_{\alpha_n}.$$

Obviously $v > 0$ on U. Given $m \in \mathbb{N}$, there exists a compact subset K of U such that $\tau_0 = w_{\alpha_1} = \cdots = w_{\alpha_m} = 0$ on $U \setminus K$ and hence $v \le \sum_{n=m+1}^\infty 2^{-n} = 2^{-m}$ on $U \setminus K$. Thus v tends to 0 at ∂U. Moreover, given $\alpha > 0$, there exists $0 < \beta < \alpha$ such that $\operatorname{dist}(x, \complement U) \le \operatorname{dist}(x, U_\alpha)$ for all $x \in U \setminus U_\beta$. Taking $m \in \mathbb{N}$ such that $\alpha_m < \beta$ we know that, for all $n \ge m$, $w_{\alpha_n} = 0$ on U_α. This shows that $v \in \mathscr{C}^\infty(U)$.

If $h \in \mathscr{H}_b(U)$ such that $h(0) = 0$ then clearly $\int_U hv\,d\lambda = 0$. Finally, we define $w = (\int_U v\,d\lambda)^{-1} v$. Then, for every $h \in \mathscr{H}_b(U)$,

$$\int_U hw\,d\lambda = \int_U (h - h(0))w\,d\lambda + \int_U h(0)w\,d\lambda = 0 + h(0)\int_U w\,d\lambda = h(0). \quad \square$$

3.3. **Corollary.** *Let* $0 < \varepsilon < 1$, *and let* $U \subset \mathbb{R}^d$ *be a bounded domain with* $\mathscr{C}^{1+\varepsilon}$*-boundary (or, more generally, with Ljapounov-Dini boundary; see* [Wi]) *such that* $0 \in U$. *Then there exists a bounded function* $w \in \mathscr{C}^\infty(U)$ *such that* $\inf w_0(U) > 0$ *and*

$$\int_U h w_0 \, d\lambda = h(0) \quad \text{for all } h \in \mathscr{H}_b(U).$$

Proof. Let $g = G_U(0, \cdot)$ and

$$U_\alpha = \{g > \alpha\}, \quad \mu_\alpha = \varepsilon_0^{\complement U_\alpha} \qquad (0 \le \alpha < \infty).$$

Note that $U_0 = U$. As proven in [Wi, p. 27] the gradient ∇g has a continuous extension to $\overline{U}$ (still denoted by ∇g) and $\|\nabla g\|$ is bounded away from zero on ∂U. Consequently, there exist $c > 0$ and $\delta > 0$ such that $\|\nabla g\| \ge c$ on $U \setminus U_\delta$. Define $w_1 : U \to \mathbb{R}$ by

$$w_1 = \begin{cases} 0 & \text{on } U_\delta, \\ \|\nabla g\|^2 \exp\left(-\frac{1}{\delta - g}\right) & \text{on } U \setminus U_\delta. \end{cases}$$

Then $w_1 \in \mathscr{C}_b^\infty(U)$ and $\inf w_1(U \setminus U_{\delta/2}) > 0$. Let σ denote the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. Recall that, for every $0 \le \alpha \le \delta$,

$$\mu_\alpha = |\langle n_\alpha, \nabla g\rangle| 1_{\partial U_\alpha} \sigma = \|\nabla g\| 1_{\partial U_\alpha} \sigma$$

(n_α exterior normal to U_α; see [Da]). Fix $h \in \mathscr{H}_b(U)$ such that $h(0) = 0$. Then

$$\begin{aligned} \int_U h w_1 \, d\lambda &= \int_{U \setminus U_\delta} h w_1 \, d\lambda \\ &= \int_0^\delta \left(\int_{\partial U_\alpha} h \|\nabla g\|^2 \exp\left(-\frac{1}{\delta - g}\right) \frac{1}{|\langle n_\alpha, \nabla g\rangle|} \, d\sigma \right) d\alpha \\ &= \int_0^\delta \left(\int_{\partial U_\alpha} h \exp\left(-\frac{1}{\delta - \alpha}\right) d\mu_\alpha \right) d\alpha \\ &= h(0) \int_0^\delta \exp\left(-\frac{1}{\delta - \alpha}\right) d\alpha = 0. \end{aligned}$$

To finish the proof it now suffices to take a bounded $\mathscr{C}^\infty$-function $w_0 > 0$ on U such that $\int_U h w_0 \, d\lambda = 0$ for every $h \in \mathscr{H}_b(U)$ with $h(0) = 0$ and to define

$$w = \left(\int_U (w_0 + w_1) \, d\lambda \right)^{-1} (w_0 + w_1).$$

References

[BH] J. Bliedtner and W. Hansen, *Potential theory—An analytic and probabilistic approach to balayage*, Universitext, Springer, Berlin, 1986.

[CD] G. Choquet and J. Deny, *Sur quelques propriétés de moyenne caractéristiques des fonctions harmoniques et polyharmoniques*, Bull. Soc. Math. France **72** (1944), 118–140.

[Da] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), 275–288.

[Fl] L. Flatto, *Fonctions with a mean value property*, J. Math. Mech. **10** (1961), 11–18.

[FL] A. Friedman and W. Littman, *Functions satisfying the mean value property*, Trans. Amer. Math. Soc. **102** (1962), 167–180.

[Ga] A. M. Garsia, *A note on the mean value property*, Trans. Amer. Math. Soc. **102** (1962), 181–186.

[Hi] M. R. Hirschfeld, *Sur les fonctions μ-harmoniques dans un espace localement compact mesuré*, C. R. Acad. Sci. Paris Sér. I. Math. **262** (1966), 174–176.

[Sm] E. Smyrnélis, *Mesures normales et fonctions harmoniques*, Bull. Sci. Math. (2) **95** (1971), 197–207.

[Wi] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. **21** (1967), 17–37.

[Za] L. Zalcman, *Mean values and differential equations*, Israel J. Math. **14** (1973), 339–353.

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