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THE STRONG LAW OF LARGE NUMBERS: A WEAK-l_2 VIEW

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ABSTRACT. Let (X_k) be a sequence of independent, centered, and square integrable real-valued random variables. To that sequence one associates

$$\forall n \in \mathbb{N}, \quad \xi_n = \|(2^{-n} X_k),\ 2^n + 1 \le k \le 2^{n+1}\|_{2,\infty}.$$

When there exists $K \ge 1$ such that

$$\sum_{n \ge 1} P^K(\xi_n > c_n) < +\infty,$$

where (c_n) is a suitable sequence of positive constants, then the strong law of large numbers holds if and only if (X_k/k) converges almost surely to 0.

The strong law of large numbers (SLLN) problem—for a sequence (X_k) of independent, centered, nonidentically distributed, real-valued random variables (r.v.)—has found a completely satisfactory solution under the Prohorov boundedness assumption:

$$\forall k \ge 1, \quad |X_k| \le k/L_2 k \quad \text{a.s.},$$

where $L_2 x = \ln \ln \sup(x, e^e)$ [15].

The hypotheses of Prohorov's result have the advantage of being stated in terms of the individual laws of the r.v. X_k and are also very easy to check in concrete situations. Nagaev's [13] general necessary and sufficient condition for the SLLN does not have this double advantage: it is a theoretical statement whose hypotheses are hard to check in practice. This fact explains why, after Nagaev gave, in 1972, his necessary and sufficient condition for the SLLN, people continued to search for sufficient (and if possible also necessary) conditions for the SLLN, conditions which would be both sharp and easy to check.

A part of this recent research is based on the nice properties of the nonincreasing rearrangement of a sequence of r.v. (X_k), for instance, the Pisier-Rodin-Semyonov theorem [14] on the weak-l_p norm $(1 < p < 2)$ of a sequence of weighted Rademacher r.v. or the bound given in [11, Theorem 3.3] for the tail of the weak-l_p norm $(p > 0)$ of a sequence of positive and independent r.v. These rearrangement properties led to the consideration of a new type of exponential inequalities—involving weak-l_p norms of the r.v.—for scalar-valued

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and vector-valued r.v. [4; 3; 5; 17, Theorem 5]. Very naturally, rearrangement techniques and weak-l_p exponential inequalities were used for studying the law of large numbers, in the general situation of vector-valued r.v. [5; 1; 9, §3].

Recently Montgomery-Smith [12] and Ledoux and Talagrand [10, Chapter 4] gave l_2 analogues of the Pisier-Rodin-Semyonov theorem. These new results also can be used as a very powerful tool for studying the SLLN on the real line.

The goal of the present paper is to obtain, via Montgomery-Smith's theorem, a scalar SLLN which unifies the main classical results and also gives a precise meaning to the informal "universal SLLN": "A sequence (X_k) of independent, centered, real-valued r.v. satisfies the SLLN if and only if it satisfies the weak law of large numbers (WLLN) and (X_k/k) converges a.s. to 0—fast enough—." Our statement (Theorem 2.1) will contain as simple corollaries Kolmogorov's SLLN [8], Prohorov's SLLN [15], and the very sharp SLLN given recently by Ledoux and Talagrand [9, Theorem 3.1].

In §1 we recall some facts on weak-l_p spaces and we state, and comment on, Montgomery-Smith's result. In §2 a general SLLN is stated (Theorem 2.1) and applied in giving a simple proof of the Ledoux and Talagrand SLLN [9, Theorem 3.1]. Finally, §3 is devoted to the proof of Theorem 2.1.

1. Some properties of Rademacher sums

Consider $p > 0$ a given number. The weak-l_p space, denoted $l_{p,\infty}$, is the only one of all sequences (a_k) of real numbers such that

$$\|(a_k)\|_{p,\infty} = \left\{\sup_{t>0} t^p \operatorname{card}(k\colon |a_k| > t)\right\}^{1/p} < +\infty. \tag{1.1}$$

It is well known that if $p > 1$, the functional $\| \ \|_{p,\infty}$ is equivalent to a norm and $l_{p,\infty}$ equipped with that norm is a Banach space. Also well known is the fact

$$\|(a_k)\|_{p,\infty} = \sup_n (n^{1/p} a_n^*), \tag{1.2}$$

where (a_n^*) denotes the nonincreasing rearrangement of the sequence $(|a_k|)$.

Let now $t > 0$ be given. Associated to t we will consider the norm $K_{1,2}(\cdot, t)$ on l_2, arising in the theory of interpolation of Banach spaces:

(1.3)

$$\forall x \in l_2, \qquad K_{1,2}(x, t) = \inf(\|x'\|_1 + t\|x''\|_2\colon x' \in l_1,\, x'' \in l_2,\, x' + x'' = x),$$

(see, for example, [7]).

Holmstedt [7, Theorem 4.1] proved the following:

Proposition 1.1. *There exists a universal constant $C_1 > 0$ such that*

(1.4)

$$\forall x \in l_2, \quad C_1^{-1} K_{1,2}(x, t) \le \sum_{1 \le k \le [t^2]} x_k^* + t\left(\sum_{k \ge [t^2]+1} (x_k^*)^2\right)^{1/2} \le K_{1,2}(x, t)$$

where $[\ ]$ stands for the integer part of a real number.

Let $x = (x_k)$ be in l_2 and (ε_k) be a sequence of i.i.d. r.v., each of them having Rademacher distribution

$$P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \tfrac{1}{2}.$$

It is well known that $X(x) = \sum \varepsilon_k x_k$ has the following tail behaviour:

$$\forall t > 0, \qquad P(X(x) > t) \le \exp(-t^2/2\|x\|_2^2). \tag{1.5}$$

Montgomery-Smith [12] has refined elementary inequality (1.5) as follows:

Proposition 1.2. *There exists a constant* $C_2 > 0$ *such that for all* $x \in l_2$ *and* $t > 0$, *one has*

$$P\{X(x) > K_{1,2}(x, t)\} \le \exp(-t^2/2) \tag{1.6}$$

and

$$P\{X(x) > (1/C_2)K_{1,2}(x, t)\} \ge (1/C_2)\exp(-C_2 t^2). \tag{1.7}$$

Remark. Results similar to inequalities (1.6) and (1.7) have also been obtained by Ledoux and Talagrand (see [10, Chapter 4]).

Our intent is to use Proposition 1.2 to prove a SLLN. More precisely, we will use (1.4) and (1.6) to bound $P((1/n)\sum_{1\le k\le n} X_k > t)$, when the r.v. X_k are independent and symmetrically distributed. To do this easily, we must first put (1.6) under a more handy form. Consider $x = (x_k)$ as an element of l_2; then, of course, x also belongs to $l_{2,\infty}$. Put $a = \|x\|_{2,\infty}$. By (1.2) and (1.4) one has

$$(1/C_1)K_{1,2}(x, t) \le a \sum_{1\le k\le [t^2]} k^{-1/2} + t\|\varphi(x, t)\|_2,$$

where $\forall k$, $\varphi_k(x, t) = x_k I(|x_k| \le a/t)$.

So finally

$$K_{1,2}(x, t) \le C_1 t(2a + \|\varphi(x, t)\|_2). \tag{1.8}$$

From this inequality one deduces the following weakened form of (1.6), which will be useful later:

Proposition 1.3. *For every* $x \in l_2$ *and every* $t > 0$, *one has*

$$P(X(x) > C_1 t(2\|x\|_{2,\infty} + \|\varphi(x, t)\|_2)) \le \exp(-t^2/2), \tag{1.9}$$

where $\forall k$, $\varphi_k(x, t) = x_k I(|x_k| \le \|x\|_{2,\infty}/t)$.

Remark. Result of a similar spirit as Proposition 1.3, but for more general r.v., appear in [4] (see for instance Theorem 1.1). Analogous exponential bounds, but for vector-valued r.v., are considered in [3] and [5].

Now we state our general SLLN.

2. A general SLLN

Let $(X_k)_{k\ge1}$ be a sequence of independent, real-valued, centered, and square integrable r.v.

For every integer n, one defines the partial sum as $S_n = \sum_{1\le k\le n} X_k$. One says that (X_k) satisfies the strong law of large numbers ($(X_k) \in$ SLLN) if $S_n/n \to 0$ a.s.; one says that (X_k) satisfies the weak law of large numbers ($(X_k) \in$ WLLN) if $S_n/n \to 0$ in probability.

The following relation between the weak and strong laws of large numbers is well known:

$$(X_k) \in \text{SLLN} \Rightarrow \text{(a) } (X_k) \in \text{WLLN}, \ \text{(b) } X_k/k \to 0 \text{ a.s.}$$

Many sufficient conditions for the SLLN are stated in the following manner: " $(X_k) \in$ WLLN and $X_k/k \to 0$ a.s. "fast enough"."

To be able to measure how fast (X_k/k) converges a.s. to 0, we need to introduce an auxiliary sequence of r.v. (ξ_n).

For every integer n, we consider the set of integers $I(n) = (2^n+1, \dots, 2^{n+1})$ and define

$$\xi_n = \|(2^{-n}X_j)_{j\in I(n)}\|_{2,\infty}.$$

We will also need the notation

$$s_n^2 = 2^{-2n} \sum_{k\in I(n)} E(X_k^2).$$

Our goal is to prove a SLLN in which the condition $X_k/k \to 0$ a.s., "fast enough" is expressed in terms of the sequence (ξ_n). That SLLN is as follows:

Theorem 2.1. *Let (X_k) be a sequence of independent, real-valued, centered, and square-integrable r.v., such that:*

(a) $X_k/k \to 0$ *a.s.*

(b) *There exists a sequence (c_n) of positive numbers such that:*

(i) $c_n/s_n \to +\infty$.

(ii) $\sum_{n\geq 1} \exp(-1/c_n^2) < +\infty$.

(iii) $\exists K > 0, \forall \varepsilon > 0, \ \sum_{n\geq 1} P^K(\xi_n > \varepsilon c_n) < +\infty$.

Then $(X_k) \in SLLN$.

To convince the reader of the strength of this result we will derive from it a rather general SLLN due to Ledoux and Talagrand:

Corollary 2.2 [9, Theorem 3.1]. *Let (X_k) be a sequence of independent, centered, square-integrable, real-valued r.v. such that $X_k/k \to 0$ a.s.*

Assume, moreover, that for some $v > 0$, all $n \in \mathbb{N}$, and $t \in]0, 1]$,

$$P\left(\sup_{k\in I(n)} |X_k| > tv2^n\right) \leq \delta_n \exp(1/t) \tag{2.1}$$

where $\sum_{n\geq 1} \delta_n^s < +\infty$, for some $s > 0$, and for all $\varepsilon > 0$

$$\sum_{n\geq 1} \exp(-\varepsilon/s_n^2) < +\infty \tag{2.2}$$

Then $(X_k) \in SLLN$.

Proof of Corollary 2.2. Let us check that the hypotheses of Theorem 2.1 are fulfilled when those of Corollary 2.2 hold. From hypothesis (2.2) it is easy to construct a sequence (a_n) of positive numbers such that $a_n/s_n \to +\infty$ and

$$\sum_{n\geq 1} \exp(-1/a_n^2) < +\infty.$$

From (a_n) one deduces (c_n) as follows:

$$c_n = \max(a_n, (s \ln(1/\delta_n))^{-1/2}).$$

Hypotheses (b)(i), (b)(ii) of Theorem 2.1 clearly hold for this sequence (c_n). Let us check that it is also the case for condition (b)(iii).

First, notice that by applying (2.1) for $t = t_n = (-2/\ln\delta_n)$, when n is large enough, one gets

$$\sum_{n\geq 1} P^{2s}\left(\sup_{k\in I(n)} |X_k| > t_n v 2^n\right) < +\infty. \tag{2.3}$$

For every integer n and every $k \in I(n)$, one defines $Y_k = X_k I(|X_k| \leq t_n v 2^n)$. In order to bound $u_n = P(\|(2^{-n}Y_k\,,\ k \in I(n))\|_{2,\infty} > \varepsilon c_n)$, we need the following general result [2, Lemma 2.16]:

Lemma 2.3. *Let* $(Z_1, \ldots, Z_n)$ *be a sequence of independent, nonnegative, r.v. such that there exists a constant* $b > 0$ *with*

$$\forall k = 1, \ldots, n, \qquad Z_k \leq b \quad a.s.$$

If one defines $A = \sup_{t>0}(t^2 \sum_{1\leq k\leq n} P(Z_k > t))$, *then one has*

$$\forall z > eA, \qquad P(\|(Z_k)\|_{2,\infty}^2 > z) \leq 1/(1 - eA/z)\exp\{-(z/b^2)\ln(z/eA)\}.$$

Denoting by A_n the quantity A associated to the sequence $(Z_k\,,\ k \in I(n))$:

$$\forall k \in I(n), \quad Z_k = 2^{-n}|Y_k|,$$

and applying Markov's inequality, one sees from the definition of c_n that (A_n/c_n^2) converges to 0. So, by Lemma 2.3, there exists a sequence $v_n = v_n(\varepsilon)$ of positive numbers, $v_n \to +\infty$, such that

$$\forall n \geq n_0, \qquad u_n \leq 2\exp(-c_n^2 v_n/t_n^2) \leq 2\exp\left(\frac{v_n}{(s\ln\delta_n)t_n^2}\right).$$

(Note that $c_n^2 \geq -\frac{1}{s\ln\delta_n}$.) So there exists n_1 such that

$$\forall n \geq n_1, \qquad u_n \leq 2\delta_n.$$

The rough estimate:

$$P(\xi_n > \varepsilon c_n) \leq P\left(\sup_{k\in I(n)} |X_k| > t_n v 2^n\right) + u_n,$$

and (2.3) therefore imply that condition (b)(iii) holds with $K = 2s$.

This ends the proof of Corollary 2.2.

Remarks. Kolmogorov's SLLN and Prohorov's SLLN, being easy consequences of Corollary 2.2, are also contained in Theorem 2.1.

One can also give another SLLN of a spirit similar to Theorem 2.1; the idea is to drop the cumbersome ε involved in (b)(iii) and, of course, to compensate this by strengthening a little bit condition (b)(ii). That SLLN, whose proof is left as an exercise to the reader is as follows:

Theorem 2.4. *Let* (X_k) *be a sequence of independent, real-valued, centered, and square-integrable r.v. such that:*

(a) $X_k/k \to 0$ *a.s.*

(b) *There exists a sequence* (c_n) *of positive numbers such that:*

(i) $\forall n \geq n_0$, $c_n^2 \geq 4s_n^2$;

(ii) $\forall \varepsilon > 0$, $\sum_{n\geq 1} \exp(-\varepsilon/c_n^2) < +\infty$;

(iii) $\exists K > 1$, $\sum_{n\geq 1} P^K(\xi_n > c_n) < +\infty$.

Then $(X_k) \in SLLN$.

3. Proof of Theorem 2.1

It follows easily from (b)(i), (ii) that $s_n^2 \to 0$; so $(X_k) \in \mathrm{WLLN}$.

As $(X_k) \in \mathrm{WLLN}$, a classical argument (see [16, p. 159]) shows that it suffices to prove Theorem 2.1 for symmetrically distributed r.v. So from now on we will suppose that the X_k are symmetrically distributed.

For every n, one defines

$$T_n = 2^{-n} \sum_{k \in I(n)} X_k. \tag{3.1}$$

The following equivalence is also well known in the symmetric case [16]:

$$(X_k) \in \mathrm{SLLN} \Leftrightarrow \forall \varepsilon > 0, \qquad \sum_{n \geq 1} P(|T_n| > \varepsilon) < +\infty. \tag{3.2}$$

To show the convergence of the series involved in equivalence (3.2), we will first use Hoffmann-Jørgensen's inequality to make it possible to apply hypothesis (b)(iii). For the reader's convenience we recall the statement of Hoffmann-Jørgensen's inequality:

Lemma 3.1 [6, (3.3), p. 164]. *Let $Y_1, \dots, Y_n$ be independent, symmetrically distributed r.v. with sum Y. Then*

$$\forall t > 0, \quad P(|Y| > 3t) \leq P\left(\sup_{1 \leq k \leq n} |Y_k| > t\right) + 4P^2(|Y| > t).$$

Lemma 3.1, applied to the r.v. $2^{-n}X_k$ when $k \in I(n)$, gives

$$\forall t > 0, \quad P(|T_n| > t) \leq P\left(\sup_{k \in I(n)} 2^{-n}|X_k| > t/3\right) + 4P^2(|T_n| > t/3).$$

Now one uses a classical iteration trick, which is to apply Lemma 3.1 again to bound $P^2(|T_n| > t/3)$.

So, if L is the smallest integer for which $2^L \geq K$, one notes that there exist two positive constants C_3 and C_4, depending only on K and such that

$$P(|T_n| > t) \leq C_3 P\left(\sup_{k \in I(n)} 2^{-n}|X_k| > 3^{-L}t\right) + C_4 P^K(|T_n| > 3^{-L}t).$$

As (X_k/k) converges a.s. to 0, it follows by the Borel-Cantelli lemma that the series with general term $P(\sup_{k \in I(n)} 2^{-n}|X_k| > 3^{-L}t)$ converges.

By hypothesis (b)(iii) it remains therefore only to check

$$\forall \varepsilon > 0, \quad \sum_{n \geq 1} P^K\left(|T_n| > C_1\varepsilon;\ \xi_n \leq \frac{\varepsilon c_n}{3\sqrt{2}}\right) < +\infty, \tag{3.3}$$

the constant C_1 being that which appears in Proposition 1.3.

For proving (3.3) we will apply Proposition 1.3 conditionally.

We first define

$$x = (2^{-n}X_k,\ k \in I(n)).$$

For every ω such that $\xi_n(\omega) \leq \varepsilon c_n/3\sqrt{2}$ and $\|\varphi(x(\omega), \sqrt{2}/c_n)\|_2 \leq \varepsilon c_n/3\sqrt{2}$ one has by Proposition 1.3

$$P'\left(\omega' : |2^{-n} \sum_{k \in I(n)} \varepsilon_k(\omega') X_k(\omega)| > C_1\varepsilon\right) \leq 2\exp(-1/c_n^2).$$

It follows that

$$
\begin{aligned}
&P\left(|T_n| > C_1\varepsilon\,;\, \xi_n \le \frac{\varepsilon c_n}{3\sqrt{2}}\right) \\
&\qquad \le 2\exp\left(\frac{-1}{c_n^2}\right) + P\left(\left\|\varphi\left(x\,,\,\frac{\sqrt{2}}{c_n}\right)\right\|_2 > \frac{\varepsilon c_n}{3\sqrt{2}}\,;\, \xi_n \le \frac{\varepsilon c_n}{3\sqrt{2}}\right).
\end{aligned} \tag{3.4}
$$

Notice that

$$
\begin{aligned}
&P\left(\left\|\varphi\left(x\,,\,\frac{\sqrt{2}}{c_n}\right)\right\|_2 > \frac{\varepsilon c_n}{3\sqrt{2}}\,;\, \xi_n \le \frac{\varepsilon c_n}{3\sqrt{2}}\right) \\
&\qquad \le P\left(2^{-2n}\sum_{k\in I(n)} X_k^2 I_{(|X_k|\le 2^n\varepsilon c_n^2/6)} > \frac{\varepsilon^2 c_n^2}{18}\right) \\
&\qquad = P\left(\exp\left\{\frac{36}{\varepsilon^2 c_n^4}2^{-2n}\sum_{k\in I(n)} X_k^2 I_{(|X_k|\le 2^n\varepsilon c_n^2/6)}\right\} > \exp\frac{2}{c_n^2}\right).
\end{aligned}
$$

By applying now Markov's inequality one gets

$$
P\left(\left\|\varphi\left(x\,,\,\frac{\sqrt{2}}{c_n}\right)\right\|_2 > \frac{\varepsilon c_n}{3\sqrt{2}}\,;\, \xi_n \le \frac{\varepsilon c_n}{3\sqrt{2}}\right) \le \exp\left(\frac{-2}{c_n^2} + \frac{36(e-1)s_n^2}{\varepsilon^2 c_n^4}\right).
$$

By assumption (b)(i) there exists n_2:

$$
\forall n \ge n_2\,,\quad P\left(\left\|\varphi\left(x\,,\,\frac{\sqrt{2}}{c_n}\right)\right\|_2 > \frac{\varepsilon c_n}{3\sqrt{2}}\,;\, \xi_n \le \frac{\varepsilon c_n}{3\sqrt{2}}\right) \le \exp\left(\frac{-1}{c_n^2}\right). \tag{3.5}
$$

Combining (3.4) and (3.5) gives property (3.3), and this concludes the proof of Theorem 2.1.

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