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THE WEAK STABILITY OF THE POSITIVE FACE IN L^1

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ABSTRACT. Let F be the positive face of the unit ball of $L^1[0, 1]$. We show that F is weakly stable in the sense that the midpoint map $\Phi_{1/2} : F \times F \to F$, with $\Phi_{1/2}(f, g) = \frac{1}{2}(f+g)$, is open with respect to the weak topology. This weak stability of the set F is the reason behind the fact that the notions of "huskable" and "strongly regular" operators coincide for operators from $L^1[0, 1]$ to a Banach space X. We prove this stability by showing that if $f_1, f_2 \in F$, $\lambda \in (0, 1)$, $\varepsilon > 0$, and $\delta \geq \max\{2\varepsilon/\lambda, 2\varepsilon/(1-\lambda)\}$, then

$$\lambda V_{P,\delta}(f_1) + (1-\lambda)V_{P,\delta}(f_2) \supset V_{P,\varepsilon}[\lambda f_1 + (1-\lambda)f_2],$$

where $P = \{A_1, \dots, A_n\}$ is a finite positive partition of $[0, 1]$ and

$$V_{P,\varepsilon}(f) = \left\{ g \in F : \sum_{i=1}^{n} \left| \int_{A_i} (f-g)(t)\, d\mu(t) \right| \leq \varepsilon \right\}$$

for any f in F. We construct an example showing that for any $0 < \lambda < 1$ there are functions f_1 and f_2 in F such that if $0 < \varepsilon < 2\min\{\lambda, 1-\lambda\}$ and $0 \leq \delta < \max\{\varepsilon/\lambda, \varepsilon/(1-\lambda)\}$, then

$$\lambda V_{P,\delta}(f_1) + (1-\lambda)V_{P,\delta}(f_2) \not\supset V_{P,\varepsilon}(\lambda f_1 + (1-\lambda)f_2).$$

Thus the "formula" that $\lambda V_{p,\varepsilon}(f_1) + (1-\lambda)V_{p,\varepsilon}(f_2) = V_{p,\varepsilon}(\lambda f_1 + (1-\lambda)f_2)$ given by Ghoussoub et al. in Mem. Amer. Math. Soc., vol. 70, no. 378, which is used there to establish the weak stability of F, is false.

Let C be a convex subset of some topological vector space X. C is said to be stable if the midpoint map $\Phi_{1/2}: C \times C \to C$, with $\Phi_{1/2}(x, y) = \frac{1}{2}(x+y)$, is open. If X is a Banach space and C is stable with respect to the weak topology, then we say that C is weakly stable. It was proved in [1, Proposition 1.1] that if C is stable and X is locally convex, then for any λ in $[0, 1]$ the map $\Phi_\lambda: C \times C \to C$, with $\Phi_\lambda(x, y) = \lambda x + (1+\lambda)y$, is also open. Note that the conclusion holds without assuming X to be locally convex. Hence convex combinations of nonempty relatively open subsets of a stable set are relatively open.

Throughout, the triple (Ω, Σ, μ) will denote the Lebesgue measure space on $[0, 1]$ and L^1 will be the Banach space of all (equivalence classes of) Lebesgue integrable functions on $[0, 1]$ equipped with the norm $\|f\|_1 = \int_\Omega |f(t)|\, d\mu(t)$.

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We shall denote by F the positive face of the unit ball of L^1, i.e.,

$$F = \{f \in L^1 : f \geq 0 \text{ and } \|f\|_1 = 1\}.$$

Let $P = \{A_1, \dots, A_n\}$ be a finite positive partition of Ω and $\varepsilon \geq 0$. Define, for $f \in F$,

$$V_{P,\varepsilon}(f) = \left\{ g \in F : \sum_{i=1}^{n} \left| \int_{A_i} (f-g)\, d\mu \right| \leq \varepsilon \right\}$$

and

$$V^0_{P,\varepsilon}(f) = \left\{ g \in F : \sum_{i=1}^{n} \left| \int_{A_i} (f-g)\, d\mu \right| < \varepsilon \right\}.$$

As pointed out in [2] the sets $V^0_{P,\varepsilon}(f)$ form a relative weak neighborhood base of f in F when P runs through the finite positive partitions of Ω and ε runs through $(0, 1]$.

The weak stability of the face F is the reason behind the fact that the notions of "huskable" and "strongly regular" operators coincide for operators from $L^1[0, 1]$ to a Banach space X (cf. [2, Theorem IV.10]). In [2], the weak stability of F is established using the following lemma.

Lemma 1 [2, Lemma IV.4]. *If* $f_1, f_2 \in F$ *and* $\lambda \in [0, 1]$, *then*

$$\lambda V_{P,\varepsilon}(f_1) + (1-\lambda) V_{P,\varepsilon}(f_2) = V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2).$$

However, the formula in this lemma need not hold as seen by the following counterexample (Example 3). Consequently, it is important to establish that F is weakly stable without using this formula. Theorem 4 gives a correct variant of the above lemma which is strong enough to conclude the desired weak stability result. We will use the following lemma in Example 3.

Lemma 2 [2, Lemma IV.3]. *For* $f \in F$ *and* $\varepsilon \geq 0$, *the following holds true:*

$$V_{P,\varepsilon}(f) = [V_{P,0}(f) + \varepsilon B_{L^1}] \cap F,$$

where B_{L^1} *is the closed unit ball of* L^1.

Example 3. Fix $\lambda \in (0, 1)$. Let $A_1 = [0, \lambda)$, $A_2 = [\lambda, 1]$, $f_1 = \frac{1}{\lambda}\chi_{A_1}$, $f_2 = \frac{1}{1-\lambda}\chi_{A_2}$, and $P = \{A_1, A_2\}$. Fix $0 < \varepsilon < 2\min\{\lambda, 1-\lambda\}$, and let $g = \frac{\varepsilon}{2\lambda}\chi_{A_1} - \frac{\varepsilon}{2(1-\lambda)}\chi_{A_2}$ and $g' = -g$. It is clear that f_1, f_2 are in F and that $\lambda f_1 + (1-\lambda) f_2 + g$ and $\lambda f_1 + (1-\lambda) f_2 + g'$ are in $V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2)$. Let $\rho = \max\{\frac{1}{\lambda}, \frac{1}{1-\lambda}\}$. The proof below shows that if $0 \leq \delta < \rho\varepsilon$ then $\lambda f_1 + (1-\lambda) f_2 + g$ or $\lambda f_1 + (1-\lambda) f_2 + g'$ does not belong to $\lambda V_{P,\delta}(f_1) + (1-\lambda) V_{P,\delta}(f_2)$. In particular, if $0 \leq \delta < \rho\varepsilon$ then

$$\lambda V_{P,\delta}(f_1) + (1-\lambda) V_{P,\delta}(f_2) \not\supset V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2).$$

Thus by taking $\delta = \varepsilon$ we see that [2, Lemma IV.4] need not hold.

Proof. Suppose that for some $\delta \geq 0$ both $\lambda f_1 + (1-\lambda) f_2 + g$ and $\lambda f_1 + (1-\lambda) f_2 + g'$ are in $\lambda V_{P,\delta}(f_1) + (1-\lambda) V_{P,\delta}(f_2)$. We will show that $\delta \geq \rho\varepsilon$.

Since $\lambda f_1 + (1-\lambda) f_2 + g \in \lambda V_{P,\delta}(f_1) + (1-\lambda) V_{P,\delta}(f_2)$, by Lemma 2 there are functions $h_i \in V_{P,0}(f_i)$ and $g_i \in \delta B_{L^1}$ such that $h_i + g_i \in V_{P,\delta}(f_i)$ and

$$\lambda f_1 + (1-\lambda) f_2 + g = \lambda (h_1 + g_1) + (1-\lambda)(h_2 + g_2).$$

Note that h_i and $h_i + g_i$ are in F and that $\int_{A_j} h_i \, d\mu = \int_{A_j} f_i \, d\mu = \delta_{ij}$ for $i, j \in \{1, 2\}$. Thus $\int_\Omega g_i \, d\mu = 0$. Furthermore, almost everywhere on A_2, we have that $h_1 = 0$ and so $g_1 \geq 0$. It follows that

$$\begin{aligned}(1-\lambda) - \frac{\varepsilon}{2} &= \int_{A_2} [\lambda f_1 + (1-\lambda) f_2 + g] \, d\mu \\ &= \int_{A_2} [\lambda(h_1 + g_1) + (1-\lambda)(h_2 + g_2)] \, d\mu \\ &= (1-\lambda) + \lambda \int_{A_2} g_1 \, d\mu + (1-\lambda) \int_{A_2} g_2 \, d\mu\end{aligned}$$

and so

$$-\int_{A_1} g_2 \, d\mu = \int_{A_2} g_2 \, d\mu = -\frac{\varepsilon}{2(1-\lambda)} - \frac{\lambda}{1-\lambda} \int_{A_2} g_1 \, d\mu \leq -\frac{\varepsilon}{2(1-\lambda)}.$$

Thus $\frac{\varepsilon}{1-\lambda} \leq \|g_2\|_1 \leq \delta$. Similarly $\frac{\varepsilon}{\lambda} \leq \|g_1\|_1 \leq \delta$. Therefore, $\delta \geq \rho\varepsilon$.

However, one can still conclude the weak stability of F from the following variant of [2, Lemma IV.4].

Theorem 4. *Let f_1, f_2 be in F. If $\lambda \in (0, 1)$, $\varepsilon > 0$, and $\delta \geq \max\{\frac{2\varepsilon}{\lambda}, \frac{2\varepsilon}{1-\lambda}\}$, then*

$$\lambda V_{P,\delta}(f_1) + (1-\lambda) V_{P,\delta}(f_2) \supset V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2).$$

Proof. Let $\{A_1 \dots, A_n\}$ be the sets of the partition P. Let $a_i = \int_{A_i} f_1 \, d\mu$ and $b_i = \int_{A_i} f_2 \, d\mu$ for $1 \leq i \leq n$. Let g be any function in $V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2)$. Put

$$\alpha_i = \frac{a_i}{\lambda a_i + (1-\lambda) b_i} \quad \text{and} \quad \beta_i = \frac{b_i}{\lambda a_i + (1-\lambda) b_i},$$

observing the convention that $\frac{0}{0}$ is 1. Note that $0 \leq \alpha_i \leq \frac{1}{\lambda}$ and $0 \leq \beta_i \leq \frac{1}{1-\lambda}$. Let

$$h_1(\cdot) = \sum_{i=1}^n \alpha_i g(\cdot) \chi_{A_i} \quad \text{and} \quad h_2(\cdot) = \sum_{i=1}^n \beta_i g(\cdot) \chi_{A_i}.$$

Clearly $h_i \geq 0$ and $\lambda h_1 + (1-\lambda) h_2 = g$. Thus, without loss of generality, $\|h_1\|_1 \geq 1$. Let

$$g_1 = \frac{h_1}{\|h_1\|_1} \quad \text{and} \quad g_2 = \frac{1}{1-\lambda} g - \frac{\lambda}{1-\lambda} g_1.$$

Clearly $g_1 \in F$ and $h_1 \geq g_1 \geq 0$ and $\lambda g_1 + (1-\lambda) g_2 = g$. Hence $\lambda(h_1 - g_1) = (1-\lambda)(g_2 - h_2)$ and so $g_2 \geq h_2 \geq 0$, thus $g_2 \in F$. To complete the proof we need only to show that $\sum_{i=1}^n |\int_{A_i} (f_j - g_j) \, d\mu| \leq \delta$ for $j = 1$ and 2.

Toward this, first note that

$$\begin{aligned}\sum_{i=1}^n \left| \int_{A_i} (f_1 - h_1) \, d\mu \right| &= \sum_{i=1}^n \left| a_i - \alpha_i \int_{A_i} g \, d\mu \right| = \sum_{i=1}^n \alpha_i \left| \lambda a_i + (1-\lambda) b_i - \int_{A_i} g \, d\mu \right| \\ &= \sum_{i=1}^n \alpha_i \left| \int_{A_i} [\lambda f_1 + (1-\lambda) f_2 - g] \, d\mu \right| \leq \frac{\varepsilon}{\lambda}.\end{aligned}$$

Likewise, $\sum_{i=1}^n |\int_{A_i}(f_2 - h_2)\,d\mu| \le \frac{\varepsilon}{1-\lambda}$. Thus

$$\begin{aligned}\sum_{i=1}^n \left|\int_{A_i}(h_1 - g_1)\,d\mu\right| &= \sum_{i=1}^n \left|\int_{A_i} h_1\left(1 - \frac{1}{\|h_1\|_1}\right)d\mu\right| = \left|1 - \frac{1}{\|h_1\|_1}\right| \|h_1\|_1 \\ &= |\,\|h_1\|_1 - 1| = \left|\int_\Omega h_1\,d\mu - \int_\Omega f_1\,d\mu\right| \\ &\le \sum_{i=1}^n \left|\int_{A_i}(f_1 - h_1)(t)\,d\mu(t)\right| \le \frac{\varepsilon}{\lambda}\end{aligned}$$

and so

$$\sum_{i=1}^n \left|\int_{A_i}(h_2 - g_2)\,d\mu\right| = \frac{\lambda}{1-\lambda}\sum_{i=1}^n \left|\int_{A_i}(h_1 - g_1)\,d\mu\right| \le \frac{\varepsilon}{1-\lambda}.$$

Hence $\sum_{i=1}^n |\int_{A_i}(f_j - g_j)\,d\mu| \le \delta$ for $j = 1$ and 2, as needed, and the proof is complete.

Corollary 5. *The positive face F is weakly stable.*

Proof. Suppose V_1 and V_2 are two nonempty relatively weakly open subsets of F. Let f_i be any point in V_i. Since the set $V^0_{P,\varepsilon}(f)$ forms a relative weak neighborhood base of f in F, when P runs through the finite positive partitions of Ω and ε runs through $(0, 1]$, there exist partitions P_1 and P_2 and positive numbers δ_1 and δ_2 such that $V^0_{P_i,\delta_i}(f_i) \subset V_i$. Let $0 < \delta < \min\{\delta_1, \delta_2\}$, $0 < \varepsilon < \frac{1}{4}\delta$, and let P be a finer partition of Ω than P_1 and P_2. By Theorem 4, we have

$$\tfrac{1}{2}V_{P,\delta}(f_1) + \tfrac{1}{2}V_{P,\delta}(f_2) \supset V_{P,\varepsilon}(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2).$$

Since

$$\begin{aligned}\tfrac{1}{2}V_1 + \tfrac{1}{2}V_2 &\supset \tfrac{1}{2}V^0_{P_1,\delta_1}(f_1) + \tfrac{1}{2}V^0_{P_2,\delta_2}(f_2) \supset \tfrac{1}{2}V_{P,\delta}(f_1) + \tfrac{1}{2}V_{P,\delta}(f_2), \\ &\supset V_{P,\varepsilon}(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2) \supset V^0_{P,\varepsilon}(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2),\end{aligned}$$

the set $\frac{1}{2}V_1 + \frac{1}{2}V_2$ is a weak neighborhood of $\frac{1}{2}f_1 + \frac{1}{2}f_2$ and so $\frac{1}{2}V_1 + \frac{1}{2}V_2$ is weakly open. Therefore, F is weakly stable, and the proof is complete.

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