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NONLINEAR DEMIREGULAR APPROXIMATION SOLVABILITY OF EQUATIONS INVOLVING STRONGLY ACCRETIVE OPERATORS

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ABSTRACT. Based on the demiregular convergence theory of the operator equations involving strongly monotone operators by Anselone and Lei (1986), we study approximation solvability of the nonlinear equations involving strongly accretive operators.

1. INTRODUCTION

In a series of publications on regular operator approximation theory—an improvement over the usual convergence methods for approximate solutions in the sense that the convergence is achieved under minimal hypotheses—Anselone and Ansorge [1] and Anselone and Lei [2] investigated and studied the problems on the solvability of nonlinear equations and its applications, especially to nonlinear integral equations of the Urysohn and Hammerstein types. Ansorge and Lei [4] studied problems on the solvability of nonlinear elliptic equations and nonlinear hyperbolic equations with interesting examples. More significantly, Anselone and Lei [3] generalized the regular operator approximation theory to the case of the demiregular convergence and applied the obtained results to the case of the fixed point and bifurcation approximations with examples. As a result of this development, existence and convergence results are derived under weaker hypotheses. They further applied this theory to the approximation solvability of the nonlinear equations involving strongly monotone operators.

Here we study, based on demiregular convergence, the approximation solvability of nonlinear equations involving strongly accretive operators. Although our main results can be achieved by restricting the duality map to weak continuity, we adopt a direct approach—a property of Reich [5] studied by Webb [7]—of bypassing the stringent hypothesis on the duality map. In a way the obtained results generalize the work of Anselone and Lei [3].

2. DEMIREGULAR CONVERGENCE

Let X and Y be real Banach spaces with corresponding dual spaces X^* and Y^*. Let $x, x_n \in X$, $y, y_n \in Y$, and $n \in N$ (the set of all natural numbers).

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The symbols " $\to$ " and " $\xrightarrow{\omega}$ " shall denote the strong convergence and weak convergence, respectively. Consider operators $A, A_n: X \to Y$ for $n \in N$.

For any $x, x_n \in X$, $n \in N$, an operator $A: X \to Y$ is said to be *demicontinuous* if $x_n \to x$ implies $Ax_n \xrightarrow{\omega} Ax$ as $n \to \infty$. An operator sequence $\{A_n\}$ is said to be (as $n \to \infty$):

asymptotically demicontinuous if $x_n \to x$ implies $A_n x_n - A_n x \xrightarrow{\omega} 0$,
demipointwise convergent $(A_n \xrightarrow{\omega} A)$ if $A_n x \xrightarrow{\omega} Ax$,
demicontinuously convergent $(A_n \xrightarrow{\mathrm{dc}} A)$ if $x_n \to x$ implies $A_n x_n \xrightarrow{\omega} Ax$.

Next, we recall the concepts of regular operators and demiregular convergence. Let $N', N'', \ldots$ denote infinite subsets of N.

A sequence $\{x_n\}$ is said to be *d-compact* if each subsequence $\{x_n: n \in N'\}$ has a strongly convergent subsequence $\{x_n : n \in N''\}$.

A sequence $\{S_n\}$ is called *d-compact* if each subsequence $\{x_n \in S_n: n \in N'\}$ has a strongly convergent subsequence $\{x_n \in S_n: n \in N''\}$. If $S_n \neq \varnothing$ for only finitely many n, then $\{S_n\}$ is trivially d-compact.

The sets
$$\{x_n\}^* = \{x \in X: x_n \to x, \ n \in N'\}$$
and
$$\{S_n\}^* = \{x \in X: x_n \to x, \ x_n \in S_n, \ n \in N'\}$$
are called the *cluster point sets.* Thus, $\{x_n\}^* \neq \varnothing$ if $\{x_n\}$ is d-compact, and $\{S_n\}^* \neq \varnothing$ for $n \in N'$ if $\{S_n\} \neq \varnothing$ for $n \in N'$ and $\{S_n\}$ is d-compact.

A set sequence $\{S_n\}$ is said to converge to a set S $(S_n \to S$ as $n \to \infty)$ if any ε-neighborhood of S contains S_n for all n sufficiently large. This limit is not unique, that is,
$$S_n \to S \subset S' \implies S_n \to S'.$$
Furthermore, $S_n = \varnothing$ for $n \in N \implies S_n \to S$ for $S \subset X$, $S_n \to \varnothing \implies S_n = \varnothing$ for n sufficiently large, and $S_n \neq \varnothing$ for $n \in N'$, $S_n \to S \implies S \neq \varnothing$.

An operator $A: X \to Y$ is called *regular* if $\{x_n\}$ bounded and $\{Ax_n\}$ d-compact implies $\{x_n\}$ d-compact.

An operator sequence $\{A_n\}$ is said to be *asymptotically regular* if $\{x_n\}$ bounded and $\{A_n x_n\}$ d-compact implies $\{x_n\}$ d-compact and every subsequence of $\{A_n\}$ has this property.

An operator sequence $\{A_n\}$ is said have the *demiregular convergence* $(A_n \xrightarrow{\mathrm{dr}} A)$ if $A_n \xrightarrow{\mathrm{dc}} A$ and $\{A_n\}$ is asymptotically regular. It is not hard to see that $A_n \xrightarrow{\mathrm{dr}} A$ implies A is demicontinuous.

Now we need to recall the following useful properties [1]:

(P1) If $A_n \xrightarrow{\mathrm{dr}} A$, $\{x_n\}$ is bounded, and $A_n x_n \to y$, then $\{x_n\}^* \neq \varnothing$ and $Ax = y$ for all $x \in \{x_n\}^*$.

(P2) $\{S_n\}$ d-compact and $\{S_n\}^* \subset S \implies S_n \to S$.

Next, we recall the following result:

Lemma 2.1 [7, Proposition]. *Let X be a separable space with X^* uniformly convex and let $\{x_n\}$ be a bounded sequence in X. Then there exist a subsequence $\{x_k\}$ and a point $x \in X$ such that $\{J(x_k - x)\} \xrightarrow{\omega} 0$ in X^*, where $J: X \to X^*$ is a normalized duality mapping.*

3. Approximation solvability

We consider the approximation solvability based on the demiregular convergence. Let X be a real Banach space and X^* its dual. For $n \in N$, let P_n be projection on X with $P_n X = X_n$, $\dim X_n < \infty$, and $P_n \to I$ as $n \to \infty$, that is, $P_n x \to x$, $x \in X$. Then $\{P_n\}$ is uniformly bounded, $P_n \xrightarrow{c} I$, that is, $P_n x_n \to x$ as $x_n \to x$, and, thus, the convergence is uniform on compact sets. Let $T: X \to X$, and let P_n^* be projection on X^* with $P_n^* \to I^*$ as $n \to \infty$, and X^* uniformly convex.

Now we consider the equations

$$(3.1) \qquad Tx = y, \quad x, y \in X, \qquad \text{and} \qquad P_n T x_n = P_n y, \quad x_n \in X_n, \ y \in X.$$

Here, since $P_n^* Jx = Jx$, $x \in X_n$, we have

$$\begin{aligned} P_n T x_n = P_n y \iff & [P_n(Tx_n - y), P_n^* Jx] \\ &= [P_n(Tx_n - y), Jx] = [Tx_n - y, P_n^* Jx] = [Tx_n - y, Jx] = 0, \end{aligned}$$

where $[\cdot, \cdot]$ is the duality pairing between X and X^*. At this point, we need to recall an alternative definition (Webb [7]) of strong accretiveness of an operator $T: X \to X$ in terms of a *normalized duality mapping* $J: X \to X^*$, that is,

$$[x, Jx] = \|x\|^2 \quad \text{and} \quad \|Jx\| = \|x\|, \qquad x \in X.$$

An operator $T: X \to X$ is called *strongly accretive* iff, for all $x, y \in D(T)$ and $\alpha > 0$,

$$(3.2) \qquad [Tx - Ty, J(x-y)] \ge \alpha \|x - y\|^2.$$

Now we are about to consider the auxiliary and main results on demiregular convergence.

Lemma 3.1. *Let $T: X \to X$ be strongly accretive with constant $\alpha > 0$. If $y \in X$ and $\gamma > 0$, then $[Tx - y, Jx] > 0$ for $\|Jx\| = \|x\| \ge \gamma > \|T(0) - y\|/\alpha$.*

Proof. We have, for $x, y \in X$,

$$\begin{aligned} [Tx - y, Jx] &= [Tx - T(0), Jx] + [T(0) - y, Jx] \\ &\ge \alpha\|x\|^2 - \|T(0) - y\| \, \|x\| > \alpha\|x\|^2 - \alpha\gamma\|x\| \ge 0. \quad \square \end{aligned}$$

Theorem 3.2. *Let $T: X \to X$ be demicontinuous with $x \in X$, $\gamma > 0$, and $[Tx - y, Jx] > 0$ for $\|x\| = \gamma$. Then $P_n T x_n = P_n y$ for some $x_n \in X_n$ with $\|x_n\| < \gamma$, $n \in N$.*

Proof. The proof follows from Lemma 3.1, and the continuity of T on finite-dimensional space X_n. $\square$

Theorem 3.3. *For fixed $y \in X$ and $\gamma > 0$, let $P_n T \xrightarrow{\text{dr}} T$ and $[Tx - y, Jx] > 0$ for $\|x\| = \gamma$. If we set*

$$S = \{x \in X : Tx = y, \ \|x\| \le \gamma\}$$

and

$$S_n = \{x_n \in X_n : P_n T x_n = P_n y, \ \|x_n\| \le \gamma\},$$

then $S_n \neq \varnothing$ *for* $n \in N$, $S_n \to S$, *and* $S \neq \varnothing$, *where* $J: X \to X^*$ *is a normalized duality mapping.*

Proof. Since $P_n T \xrightarrow{\text{dr}} T$ implies T is demicontinuous, and since $[Tx - y, Jx] > 0$ for $\|x\| = \gamma$, we obtain from Theorem 3.2 that $P_n T x_n = P_n y$ for $x_n \in X_n$ with $\|x_n\| < \gamma$, $n \in N$, and consequently, $S_n \neq \varnothing$ for $n \in N$. Since $\bigcup X_n$ is bounded, so is $\bigcup S_n$. Given that $\{P_n A\}$ is asymptotically regular and $P_n y \to y$, it follows that $\{S_n\}$ is d-compact. Next, to show $\{S_n\}^* \subset S$, if $x \in \{S_n\}^*$ then there exist N' and $x_n \in X_n$ for $n \in N'$ such that $P_n T x_n = P_n y$ and $x_n \to x$ for $n \in N'$. Then, by property (P1), $Tx = y$. Since $\{X_n\}^* \subset X$, it follows that $x \in S$. Finally, in view of property (P2), $S_n \to S$. □

Theorem 3.4. *Let* $T: X \to X$ *be bounded, demicontinuous, and strongly accretive with a constant* $\alpha > 0$. *If* $y \in X$ *and* $\gamma > \|T(0) - y\|/\alpha$, *then* $P_n T \xrightarrow{\text{dr}} T$, *and the equation* $Tx = y$ *has a unique solution* x *with* $\|x\| \leq \gamma$. *Furthermore, if*

$$S_n = \{x_n \in X_n : P_n T x_n = P_n y, \ \|x_n\| \leq \gamma\},$$

then $S_n \neq \varnothing$ *for* $n \in N$ *and* $S_n \to \{x\}$.

Corollary 3.5. *Let* $y \in X$ *and* $\gamma > \|T(0) - y\|/\alpha$. *If* $T: X \to X$ *is bounded, continuous, and strongly accretive with a constant* $\alpha > 0$, *and if* $\gamma > \|T(0) - y\|/\alpha$ *for* $y \in X$, *then* $P_n T \xrightarrow{\text{r}} T$ *and the equation* $Tx = y$ *has a unique solution* x *with* $\|x\| \leq \gamma$. *In addition, if*

$$S_n = \{x_n \in X_n : P_n T x_n = P_n y, \ \|x_n\| \leq \gamma\}.$$

then $S_n \neq \varnothing$ *for* $n \in N$ *and* $S_n \to \{x\}$.

Proof of Theorem 3.4. We first prove $P_n T \xrightarrow{\text{dr}} T$. Since T is bounded and demicontinuous, this implies that $P_n T \xrightarrow{\text{dc}} T$. Assume $\{x_n\}$ is bounded and $\{P_n T x_n\}$ is d-compact. Then $\{T x_n\}$ is bounded, $P_n T x_n \to y$ for some $y \in X$, $n \in N''$, and, by Lemma 2.1, $J(x_n - x) \xrightarrow{\omega} 0$ for some $x \in X$, $n \in N'$, $N'' \subset N' \subset N$. Now, $P_n \to I$ and $P_n x_n = x_n$. Since T is strongly accretive and since $P_n^* J u = J u$ for $u \in X_n$, we have, for $n \in N''$,

$$\begin{aligned}
\alpha \|x_n - x\|^2 &\leq [T x_n - T x, J(x_n - x)] \\
&= [T x_n, J(x_n - x)] - [T x, J(x_n - x)] \\
&= [T x_n, P_n^* J(x_n - x)] - [T x, J(x_n - x)] \\
&= [P_n T x_n, J(x_n - x)] - [T x, J(x_n - x)] \to 0.
\end{aligned}$$

This also implies that $x_n \to x$ for $n \in N''$ and, thus, $\{x_n\}$ is d-compact. Therefore, $\{P_n T\}$ is asymptotically regular.

The proof of the second part is as follows: From Lemma 3.1 and Theorem 3.3, it follows that $S_n \neq \varnothing$ for $n \in N$. Since T is strongly accretive, this implies from inequality (3.2) that the equation $Tx = y$ has a unique solution x and that $S_n \to \{x\}$. □

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