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BEST APPROXIMATION IN $L^1(I, X)$

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ABSTRACT. Let X be a Banach space and G a closed subspace of X. The subspace G is called proximinal in X if for every $x \in X$ there exists at least one $y \in G$ such that $\|x - y\| = d(x, G) = \inf\{\|x - z\|: z \in G\}$.

It is an open problem whether $L^1(I, G)$ is proximinal in $L^1(I, X)$ if G is proximinal in X, where I is the unit interval with the Lebesgue measure.

In this paper, we prove the proximinality of $L^1(I, G)$ in $L^1(I, X)$ for a class of proximinal subspaces G in X.

INTRODUCTION

Let X be a Banach space and G a subspace of X. We call G proximinal in X if for every $x \in X$ there exists $y \in G$ such that

$$\|x - y\| = d(x, G) = \inf\{\|x - z\|: z \in G\}.$$

This is equivalent to $\|x - y\| \le \|x - z\|$ for all $z \in G$. It is not difficult to see that if G is proximinal, then G is closed.

Now, let I be the unit interval with the Lebesgue measure. Then, for $1 \le p < \infty$, $L^P(I, X)$ is the space of Bochner p-integrable functions (equivalence classes) defined on I with values in X. It is known that $L^P(I, X)$ is a Banach space with

$$\|f\| = \left(\int_0^1 \|f(t)\|^P \, dt\right)^{1/p}.$$

We refer to [4] for more on Bochner integrable functions. If G is a closed subspace of X, then $L^P(I, G)$ is a closed subspace of $L^p(I, X)$.

The proximinality of $L^1(I, G)$ in $L^1(I, X)$ is a very interesting problem in approximation theory. Many results have been obtained in that direction [1, 2, 4, 5, 7]. The main problem that these papers address is: "If G is proximinal in X, is $L^1(I, G)$ proximinal in $L^1(I, X)$?"

The object of this paper is to prove the proximinality of $L^1(I, G)$ in $L^1(I, X)$ when G satisifes some conditions.

Throughout this paper, if X is a Banach space, then X^* denotes the dual of X. The set of reals is denoted by R. If E is a subset of R, then 1_E denotes the characteristic function of E.

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1. Proximinality with conditions on G

Throughout this section, G is assumed to be proximinal in X. For $x \in X$, we set

$$P(x, G) = \{z \in G : \|x - z\| = d(x, G)\}.$$

The map

$$\pi : X \to 2^G, \qquad \pi(x) = P(x, G),$$

is called the proximity map of X onto G. The set $\pi^{-1}\{0\}$ will be denoted by $\widehat{G}$. Since G is proximinal, then one immediately gets that $X = G + \widehat{G}$ [4]. If π is single valued, then G is called Chebychev. For $\hat{x} \in \widehat{G}$ we set

$$\widehat{B}(\hat{x}, r) = \{\hat{y} \in \widehat{G} : \|\hat{x} - \hat{y}\| \le r\}, \qquad W(\hat{x}, r) = \widehat{B}(\hat{x}, r) + G,$$

and

$$V(\hat{x}, r) = \{x \in X : d(x, \hat{x} + G) \le r\}.$$

Now to state the main result of this section, we introduce the following definition.

Definition 1.1. The subspace G is called proximinally null compact if for every $\hat{x} \in \widehat{G}$ and every sequence $(\hat{z}_n)$ in $\widehat{G}$, satisfying $\lim_n \hat{z}_n = 0$ and $d(\hat{x} + \hat{z}_n, G) = d(\hat{x}, G)$, there exists a sequence $z_n \in P(\hat{x} + \hat{z}_n, G)$ such that $\lim_n z_n = 0$.

Clearly every finite-dimensional subspace G of X is proximinally null compact.

Now, we are ready to state the main result of this section.

Theorem 1.1. *Let G be a proximinally null compact subspace of the Banach space X. If G or $\widehat{G}$ is separable, then $L^1(I, G)$ is proximinal in $L^1(I, X)$.*

Proof. Since the proof of the theorem is a little long, we divide the proof into several steps.

Step 1. For $\hat{x} \in \widehat{G}$ and $r > 0$ there exists $\delta > 0$ such that $V(\hat{x}, \delta) \subset W(\hat{x}, r)$.

Proof. We consider two cases:

Case (i): $\hat{x} = 0$. Let $x \in V(0, r)$. Then by definition $d(x, G) \le r$. If $y \in P(x, G)$, then $x - y \in \widehat{G}$ and $\|x - y\| = d(x - y, G) = d(x, G) \le r$. Hence $x - y \in \widehat{B}(0, r)$. Then $x = y + (x - y) \in \widehat{B}(0, r) + G = W(0, r)$. So $V(0, r) \subset W(0, r)$.

Case (ii): $\hat{x} \neq 0$. If possible, assume that the claim is not true. Hence there exists $r > 0$ such that for all $\delta > 0$, $V(\hat{x}, \delta)$ is not a subset of $W(\hat{x}, r)$. Hence, for any sequence (δ_n) with $\delta_n > 0$, there exists a sequence (x_n) in X such that $x_n \in V(\hat{x}, \delta_n)$ and $x_n \notin W(\hat{x}, r)$ for all n. Thus, choosing $\delta_n \underset{n}{\to} 0$, we have

$$\lim_n d(x_n, \hat{x} + G) = 0. \tag{1}$$

Since G is proximinal in X and $d(x_n, \hat{x} + G) = d(x_n - \hat{x}, G)$, there exists a sequence (y_n) in G such that $\|x_n - \hat{x} - y_n\| = d(x_n, \hat{x} + G)$. Hence by (1) we get

$$\lim_n \|x_n - \hat{x} - y_n\| = 0. \tag{2}$$

It follows, since $d(\cdot, G)$ is a continuous function, that $d(x_n - y_n, G) \to d(\hat{x}, G) = \|\hat{x}\|$ and, consequently,

$$\lim_n \frac{d(x_n - y_n, G)}{\|\hat{x}\|} = 1. \tag{3}$$

Let

$$z_n = \frac{\|\hat{x}\|}{d(x_n - y_n, G)}(x_n - y_n).$$

Then, from (2) and (3) and the fact that $d(\lambda x, G) = \lambda d(x, G)$ for all $x \in X$ and $\lambda \geq 0$, we have

$$\lim_n \|z_n - \hat{x}\| = 0 \quad \text{and} \quad d(z_n, G) = \|\hat{x}\| \tag{4}$$

for all n. Consequently, $z_n - \hat{x} = \theta_n$, where $\theta_n \in X$ and $\lim_n \|\theta_n\| = 0$. Let $\omega_n \in P(\theta_n, G)$. Then $\hat{\omega}_n = \theta_n - \omega_n \in \widehat{G}$ for all n. Further, since $\|\theta_n\| \underset{n}{\to} 0$, we have

$$z_n - \omega_n = \hat{x} + \hat{\omega}_n, \quad \text{with } \|\hat{\omega}_n\| \underset{n}{\to} 0 \text{ and } \|\omega_n\| \underset{n}{\to} 0.$$

Now, it follows, from (4) and since $\omega_n \in G$, that

$$d(\hat{x} + \hat{\omega}_n, G) = d(z_n, G) = \|\hat{x}\| \tag{5}$$

for all n. On the other hand we have, since $\omega_n \in G$,

$$P(\hat{x} + \hat{\omega}_n, G) = P(z_n - \omega_n, G) = P(z_n, G) - \omega_n.$$

This, together with (5) and the fact that $\hat{\omega}_n \underset{n}{\to} 0$, implies that, since G is proximinally null compact,

$$d(0, P(z_n, G) - \omega_n) \underset{n}{\to} 0. \tag{6}$$

By the definition of z_n and the fact that $P(\lambda x, G) = \lambda P(x, G)$ for all $x \in X$ and $\lambda \in R$, we have

$$P(z_n, G) = \frac{\|\hat{x}\|}{d(x_n - y_n, G)} P(x_n - y_n, G).$$

The latter, together with (3) and (6) and the fact that $\omega_n \underset{n}{\to} 0$, implies that $d(0, P(x_n - y_n, G)) \underset{n}{\to} 0$. Consequently, since $y_n \in G$ (hence $P(x_n - y_n, G) = P(x_n, G) - y_n$), we get that $d(y_n, P(x_n, G)) \underset{n}{\to} 0$. Thus there exists $y'_n \in P(x_n, G)$ such that $\|y_n - y'_n\| \underset{n}{\to} 0$ and $x_n - y'_n \in \widehat{G}$ for all n. Together with (2), this implies that

$$\|x_n - y'_n - \hat{x}\| \leq (\|x_n - \hat{x} - y_n\| + \|y_n - y'_n\|) \underset{n}{\to} 0.$$

But $x_n - y'_n \in \widehat{G}$. Hence $x_n - y'_n \in \widehat{B}(\hat{x}, r)$ for large values of n. Consequently

$$x_n = (x_n - y'_n) + y'_n \in \widehat{B}(\hat{x}, r) + G = W(\hat{x}, r)$$

for large values of n. This contradicts the assumption on (x_n) that $x_n \notin W(\hat{x}, r)$ for all n. Hence the claim of Step 1 must be true.

Step 2. $V(\hat{x}, \delta)$ is closed in X for all $\hat{x} \in \widehat{G}$ and $\delta > 0$.

Proof. The claim follows from the definition of $V(\hat{x}, \delta)$ and the continuity of the function $\tau: X \to R$, $\tau(x) = d(x, \hat{x} + G)$.

Step 3. *Weakly measurable set-valued functions.* Let $f \in L^1(I, X)$. We define the set-valued map

$$\phi: I \to 2^{\widehat{G}}, \qquad \phi(t) = f(t) - P(f(t), G) = \{\hat{x} \in \widehat{G}: (f(t) - \hat{x}) \in G\}.$$

We claim that if $\widehat{G}$ is separable then ϕ is weakly measurable in the sense that $\phi^{-1}(0)$ is measurable for any open set 0 in $\widehat{G}$. Indeed,

$$\begin{aligned}\phi^{-1}(0) &= \{t: \phi(t) \cap 0 \neq \varnothing\} = \{t: \{\hat{x} \in \widehat{G}: (f(t) - \hat{x}) \in G\} \cap 0 \neq \varnothing\} \\ &= \{t: f(t) \in 0 \cap \widehat{G} + G\}.\end{aligned}$$

Hence

$$\text{(Q)} \qquad \phi^{-1}(0) = f^{-1}(0, \cap \widehat{G} + G).$$

The separability of $\widehat{G}$ gives a sequence $(\hat{y}_n)$ in $0 \cap \widehat{G}$ such that

$$\text{(7)} \qquad \{\hat{y}_n\} \text{ is dense in } 0 \cap \widehat{G}.$$

For each n, there exists (since 0 is open) $r_n > 0$ such that $\widehat{B}(\hat{y}_n, r_n) \subset 0 \cap \widehat{G}$. This, together with (7), implies that

$$0 \cap \widehat{G} = \bigcup_{n=1}^{\infty} \widehat{B}(\hat{y}_n, \alpha_n) = \bigcup_{n=1}^{\infty} \widehat{B}(\hat{y}_n, r_n)$$

for every sequence (α_n) with $0 < \alpha_n \leq r_n$. Consequently we obtain, since

$$\bigcup_{n=1}^{\infty} \widehat{B}(\hat{y}_n, \delta) + G = \bigcup_{n=1}^{\infty} W(\hat{y}_n, \delta) \quad \text{for all } \delta > 0,$$

that

$$\text{(8)} \qquad 0 \cap \widehat{G} + G = \bigcup_{n=1}^{\infty} W(\hat{y}_n, \alpha_n) = \bigcup_{n=1}^{\infty} W(\hat{y}_n, r_n)$$

for every sequence (α_n) with $0 < \alpha_n \leq r_n$. By Step 1 and the fact that

$$\text{(9)} \qquad W(\hat{y}_n, \delta) \subset V(\hat{y}_n, \delta) \quad \text{for all } n \text{ and all } \delta > 0,$$

there exists a sequence (δ_n), $\delta_n > 0$ for all n, such that

$$\text{(10)} \qquad W(\hat{y}_n, \delta_n) \subset V(\hat{y}_n, \delta_n) \subset W(\hat{y}_n, r_n) \quad \text{for all } n.$$

Replacing δ_n by $\min\{\delta_n, r_n\}$ if necessary, we may assume (since $V(\hat{y}_n, \min\{\delta_n, r_n\}) \subset V(\hat{y}_n, \delta_n)$) that $0 < \delta_n \leq r_n$ for all n. This, together with (8)–(10), implies that

$$0 \cap \widehat{G} + G = \bigcup_{n=1}^{\infty} W(\hat{y}_n, \delta_n) \subset \bigcup_{n=1}^{\infty} V(\hat{y}_n, \delta_n) \subset \bigcup_{n=1}^{\infty} W(\hat{y}_n, r_n) = 0 \cap \widehat{G} + G.$$

Hence, since each $V(\hat{y}_n, \delta_n)$ is closed and f is measurable, we get, by (Q), and $\phi^{-1}(0)$ is measurable. Hence ϕ is weakly measurable.

Step 4. $L^1(I, G)$ *is proximinal in* $L^1(I, X)$.

Proof. Let $L^1(I, \widehat{G}) = \{h \in L^1(I, X) \colon h(t) \in \widehat{G} \text{ a.e. } t\}$. If $h \in L^1(I, \widehat{G})$, then

$$\|h\| = \int_0^1 \|h(t)\|\, dt \le \int_0^1 \|h(t) - g(t)\|\, dt$$

for all $g \in L^1(I, G)$. Hence $0 \in P(h, L^1(I, G))$. Hence, as proven in [4] but straightforward to show, to prove the proximinality of $L^1(I, G)$ it is enough to prove that

$$L^1(I, X) = L^1(I, G) + L^1(I, \widehat{G}).$$

Now, if $\widehat{G}$ is separable and $f \in L^1(I, X)$, then by Step 3 the set-valued map ϕ associated with f is a weakly measurable map. Hence by Theorem 11.16 in [8, p. 133], ϕ has a measurable selection. But $\phi(t) = f(t) - P(f(t), G)$. Thus there exists a measurable function $\hat{g} \colon I \to \widehat{G}$ such that $\hat{g}(t) \in f(t) - P(f(t), G)$. This implies that $\hat{g} \in L^1(I, \widehat{G})$. Consequently, $g = f - \hat{g}$ is measurable and $f = g + \hat{g}$. Further, $g(t) \in P(f(t), G)$. So $\|g(t)\| \le 2\|f(t)\|$. Hence $g \in L^1(I, G)$, and

$$(*) \qquad L^1(I, X) = L^1(I, G) + L^1(I, \widehat{G}).$$

Finally, if G is separable and $f \in L^1(I, X)$, then Range(f) is essentially separable [8, p. 115]. Hence $X_2 = \overline{G + X_1}$ is separable, where $X_1 = \overline{\text{span(Range}(f))}$. This implies that $\widehat{G}_2 = \widehat{G} \cap X_2$ is separable. Since $X_1 \subset X_2$, then $f \in L^1(I, X_2)$. Hence, from $(*)$, we get that $L^1(I, X_2) = L^1(I, G) + L^1(I, \widehat{G}_2)$ and, consequently,

$$f \in L^1(I, G) + L^1(I, \widehat{G}_2) \subset L^1(I, G) + L^1(I, \widehat{G}).$$

This completes the proof of the theorem.

2. Proximinality with conditions on $\widehat{G}$

In [5] it is shown that if G is reflexive then $L^1(I, G)$ is proximinal in $L^1(I, X)$. In this section we show that the conclusion still holds true when the reflexivity condition is on $\widehat{G}$ rather than G. More precisely we have

Theorem 2.1. *Let G be a proximinal subspace of X. If $\overline{\text{span}(\widehat{G})}$ is reflexive, then $L^1(I, G)$ is proximinal in $L^1(I, X)$.*

Proof. Let $S(X)$, $S(G)$, and $S(\widehat{G})$ be the class of simple functions with values in X, G, and $\widehat{G}$, respectively. For $u = \sum_{i=1}^n 1_{E_i} \otimes x_i \in S(X)$, let

$$v = \sum_{i=1}^n 1_{E_i} \otimes y_i \quad \text{and} \quad w = \sum_{i=1}^n 1_{E_i} \otimes w_i,$$

where $y_i \in P(x_i, G)$ and $w_i = x_i - y_i$. Then $v \in S(G)$ and $w \in S(\widehat{G})$. Further, if $E_i \cap E_j = \varnothing$ for $i \ne j$, then one can easily obtain [8] that

$$(**) \qquad v \in P(u, L^1(I, G)), \qquad w \in L^1(I, \widehat{G}), \quad \text{and} \quad u = v + w.$$

Now let $f \in L^1(I, X)$ and (f_n) be a sequence in $S(X)$ such that $\|f_n - f\| \underset{n}{\to} 0$. Choose (f_n) such that $\|f_n(t) - f(t)\| \underset{n}{\to} 0$, a.e. t. Then, by $(**)$,

$f_n = g_n + h_n$, with $g_n \in L^1(I, G)$ and $h_n \in L^1(I, \widehat{G})$ (g_n and h_n simple). Further,

$$\|h_n\| = \|f_n - g_n\| \leq \|f_n\|$$

and

$$\|h_n(t)\| = \|f_n(t) - g_n(t)\| \leq \|f_n(t)\| \quad \text{for all } t \in I.$$

It follows, since (f_n) is uniformly integrable (i.e.,

$$\sup\left\{\sup_n \left\{\int_E \|f_n(t)\|\, dt\right\} : E \subset I,\, m(E) \leq \varepsilon\right\} \to 0 \quad \text{as } \varepsilon \to 0,$$

where m denotes the Lebesgue measure) and $f_n \to f$, that (h_n) is bounded and uniformly integrable. Hence, since $h_n \in L^1(I, \widehat{G}) \subset L^1(I, \operatorname{span}(\widehat{G}))$ and $\operatorname{span}(\widehat{G})$ is reflexive, we obtain, by Dunford's Theorem [3, p. 101], that the set $\{h_n\}$ is relatively weakly compact in $L^1(I, X)$. Therefore, there exists a subsequence of (h_n) which converges weakly to some $h \in L^1(I, X)$. We will assume that $h_n \underset{n}{\to} h$ weakly. Since $f_n \underset{n}{\to} f$ in $L^1(I, X)$ and hence in the weak topology on $L^1(I, X)$, we get that (g_n) is a weakly convergent sequence, and so, there exists $g \in L^1(I, X)$ such that $g_n \underset{n}{\to} g$ weakly.

Now, we claim that $g \in L^1(I, G)$. Indeed: Since $g_n \underset{n}{\to} g$ weakly and $L^1(I, G)$ is convex [6, p. 65], there exists a sequence $(\hat{h}_n)$ in $L^1(I, G)$ such that each $\hat{h}_n$ is a finite convex combination of elements of (g_n) and $\|\hat{h}_n - g\| \underset{n}{\to} 0$. Thus $g \in L^1(I, G)$. We also claim that $g \in P(f, L^1(1, G))$. Indeed, we have

$$\begin{aligned}\|f - g\| &\leq \liminf_n \|f_n - g_n\| \qquad (\|\ \| \text{ is wkly l.s.c.})\\ &\leq \liminf_n \|f_n - w\| \quad \text{for all } w \in L^1(I, G)\ (g_n \in P(f_n, L^1(I, G)))\\ &= \|f - w\|\end{aligned}$$

for all $w \in L^1(I, G)$. Hence g is a best approximant of f in $L^1(I, G)$. This ends the proof of the theorem.

As a corollary we get the following known result [9].

Theorem 2.2. *If G is of codimension one, then the Chebychevity of G in X implies the proximinality of $L^1(I, G)$ in $L^1(I, X)$.*

Proof. Since G is Chebychev and of codimension one, it easily follows that [9] $\widehat{G}$ is a subspace of dimension one. Hence $\widehat{G}$ is reflexive. The result now follows from Theorem 2.1.

3. Further results

We present in this section some further results on the proximinality of $L^1(I, G)$ in $L^1(I, X)$.

Theorem 3.1. *Let G be a separable quasireflexive subspace of X. If G is proximinal, then $L^1(I, G)$ is proximinal in $L^1(I, X)$.*

Proof. By Theorem 1.1 in [1], it is enough to prove the result in $L^2(I, X)$. Since G is quasireflexive, then $G = F^*$ for some Banach space F. Therefore, since G is separable, G has the Radon-Nikodym property [3]. Hence [3],

$$[L^2(I, F)]^* = L^2(I, F^*) = L^2(I, G).$$

Now, let $f \in L^2(I, X)$ and f_n be a sequence of simple functions in $L^2(I, X)$ such that $\|f_n - f\|_2 \to 0$ and $\|f_n(t) - f(t)\| \to 0$ for a.e. t. As in the proof of Theorem 2.1, we let $\hat{f}_n$ be the simple function in $L^2(I, G)$ which is the best approximant of f_n. Then one easily gets $\|\hat{f}_n\| \le 2\|f_n\|$ [9], and, since $f_n \underset{n}{\to} f$ in $L^2(I, X)$, we obtain that $(\hat{f}_n)$ is a bounded sequence. But G is separable. Hence F and consequently $L^2(I, F)$ are separable. By Helly's selection theorem [6], the sequence $(\hat{f}_n)$ has a subsequence, $(\hat{f}_n)$ say, that converges to some $\hat{f}$ in $L^2(I, G)$ with the w^*-topology. We claim that $\hat{f}$ is the best approximant of f. Indeed, for any $\phi \in L^2(I, F)$ with $\|\phi\| = 1$,

$$\begin{aligned}|\langle f - \hat{f}, \phi\rangle| &= \lim_n |\langle f_n - \hat{f}_n, \phi\rangle| \le \underline{\lim}_n \|f_n - \hat{f}_n\| \\ &\le \underline{\lim}_n \|f_n - g\| \quad \text{for all } g \in L^2(I, G) \le \|f - g\|.\end{aligned}$$

Consequently $\|f - \hat{f}\| \le \|f - g\|$ for all $g \in L^2(I, G)$. This ends the proof.

The proximity map $\pi: X \to G$ is said to be weakly continuous if, whenever $\lim_n x_n = x$ in X, there exists $y_n \in P(x_n, G)$ such that y_n converges weakly to some $y \in P(x, G)$.

Theorem 3.2. *Let G be proximinal in X. If π is weakly continuous, then $L^1(I, G)$ is proximinal in $L^1(I, X)$ whenever G is separable.*

Proof. Let $f \in L^1(I, X)$ and (f_n) be a sequence of simple functions such that $\|f_n - f\| \to 0$ and $\|f_n(t) - f(t)\| \to 0$ a.e. Then, since π is weakly continuous, there exists $g_n: I \to G$ and $g: I \to G$ such that $g_n(t) \in P(f_n(t), G)$, $g(t) \in P(f(t), G)$, and, for a.e. $t \in I$,

$$\langle g_n(t), x^*\rangle \to \langle g(t), x^*\rangle \quad \text{for all } x^* \in X^*.$$

Hence, since $\langle g_n(t), x^*\rangle$ is measurable (g_n is a simple function) for each $x^* \in X^*$, we obtain that $\langle g(t), x^*\rangle$ is measurable for each $x^* \in X^*$. Therefore, by [3, p. 42] and since G is separable, we obtain that g is measurable. Further, since $g(t) \in P(f(t), G)$ for all $t \in I$, we easily obtain [9] that $\|g(t)\| \le 2\|f(t)\|$ for all $t \in I$. Therefore, $g \in L^1(I, G)$. Hence, since $g(t) \in P(f(t), G)$ for all $t \in I$, it follows immediately [8, Corollary 2.11] that $g \in P(f, L^1(I, X))$. This ends the proof.

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