TRACES AND SUBADDITIVE MEASURES ON PROJECTIONS IN JBW-ALGEBRAS AND VON NEUMANN ALGEBRAS

L. J. BUNCE AND J. HAMHALTER

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Abstract. Let $P(M)$ be the projection lattice of an arbitrary JBW-algebra or von Neumann algebra $M$. It is shown that the tracial states of $M$ correspond by extension precisely to the subadditive probability measures on $P(M)$. The analogous result for normal semifinite traces is also proved.

A positive measure on the projection lattice $P(M)$ of a JBW-algebra $M$ is a function

$$\mu: P(M) \to [0, \infty]$$

such that $\mu(e + f) = \mu(e) + \mu(f)$ whenever $ef = 0$.

Such a measure is said to be a probability measure if $\mu(1) = 1$. A positive measure is said to be normal if $\mu(e_\alpha) \to \mu(e)$ whenever $e_\alpha \not\to e$ and semifinite if for each projection $e$ there is a net of projections $e_\alpha \not\to e$ with $\mu(e_\alpha) < \infty$, for all $\alpha$.

A subadditive measure on $P(M)$ is a positive measure such that

$$\mu(e \lor f) \leq \mu(e) + \mu(f), \quad \text{for all } e, f \in P(M).$$

By the generalisation of Gleason's Theorem [3, 4] each probability measure on $P(M)$ extends uniquely to a (linear) state on $M$ whenever $M$ has no Type I$_2$ direct summand. But this theorem is false for Type I$_2$ JBW-algebras.

The aim of this paper is to show that, for every JBW-algebra $M$, each subadditive probability measure on $P(M)$ extends uniquely to a tracial state on $M$. A corollary is that each subadditive normal semifinite measure on $P(M)$ extends to a normal semifinite trace on $M$.

In particular, this involves showing that, for subadditive measures, Gleason's Theorem is true for all JBW-algebras (including Type I$_2$ JBW-algebras).

We note that traces on JBW-algebras have been extensively studied in the literature [1, 2, 5, 7, 9] and have particular significance for the criterion of types. Our results have the particular virtue of providing a purely lattice- and measure-theoretic characterisation of traces on JBW-algebras.

Since the selfadjoint part of a von Neumann algebra is a JBW-algebra, we note that our results hold for all von Neumann algebras as well as for all JBW-algebras.

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The main reference for the theory of JBW-algebras is [6], to which the reader is referred for any unmentioned details. In order to facilitate the present discussion we recall that a JBW-algebra is a real Banach dual space and that a Jordan algebra \((M, \circ)\) satisfies
\[
\|x \circ y\| \leq \|x\| \|y\| \quad \text{and} \quad \|x\| \leq \|x^2 + y^2\|.
\]
A JW-algebra is a JBW-algebra that can be realized as a weakly closed Jordan algebra of selfadjoint operators on a Hilbert space. The Type \(I_2\) JBW-algebras fall into this class and are \(L^\infty\)-sums of algebras of the form \(L^\infty(\Omega, \mu, V)\), where \(V\) is a spin factor and \(\mu\) is a Radon measure on a locally compact Hausdorff space \(\Omega\) [10].

A trace on a JBW-algebra \(M\) is a mapping \(\tau: M^+ \to [0, \infty]\) for which
\begin{enumerate}
  \item \(\tau(x + y) = \tau(x) + \tau(y)\), for all \(x, y \in M^+\).
  \item \(\tau(\lambda x) = \lambda \tau(x)\), for all \(x \in M^+\) and \(\lambda \geq 0\).
  \item \(\tau(U_x(y^2)) = \tau(U_y(x^2))\), for all \(x, y \in M\).
\end{enumerate}

The notation used in (iii) is explained as follows. For each \(x \in M\), \(U_x: M \to M\) is the positive map defined by \(U_x(y) = 2\tau_0(x_0 y) - x^2 y\). (When \(M\) is a JW-algebra with product \(x_0 y = \frac{1}{2}(xy + yx)\), this reduces to \(U_x(y) =xyx\).) If \(\tau(1) = 1\), then \(\tau\) extends uniquely to a tracial state of \(M\).

Let \(e\) and \(f\) be projections in a JBW-algebra \(M\), and let \(\tau\) be a trace on \(M\). By [5, Lemma 2.3] together with [11, Corollary 8] there is a symmetry \(s\) in \(M\) with
\[
U_s(e \vee f - f) = e - e \wedge f,
\]
from which it follows easily that \(\tau(e \vee f) \leq \tau(e) + \tau(f)\). Hence, every trace on \(M\) restricts to a subadditive measure on \(P(M)\).

**Theorem.** Let \(M\) be a JBW-algebra, and let \(\mu\) be a subadditive probability measure on \(P(M)\). Then \(\mu\) extends uniquely to a tracial state on \(M\).

**Proof.** Being a probability measure, \(\mu\) extends uniquely to a quasi-linear functional \(\bar{\mu}\) on \(M\) [3, 4]. That is, \(\bar{\mu}\) is linear (and continuous) on all abelian subalgebras of \(M\).

(i) Let \(M\) be a Type \(I_2\) JBW-algebra. Let \(x, y \in M\). We have to show that
\begin{enumerate}
  \item \(\bar{\mu}(x + y) = \bar{\mu}(x) + \bar{\mu}(y)\) and
  \item \(\bar{\mu}(xy^2 x) = \bar{\mu}(yx^2 y)\).
\end{enumerate}

Since the JBW-algebra, \(M(x, y)\), of \(M\) generated by \(x\) and \(y\) is a direct sum of an abelian algebra and a Type \(I_2\) algebra, no harm can come from supposing that \(M = M(x, y)\). In which case, let \(\pi\) be a factor representation of \(M\). Then \(\pi: M \to V\) where \(V\) is a spin factor and \(\pi(M) = V\). So
\[
\pi(x) = a1 + \beta e, \quad \pi(y) = \lambda l + \mu f
\]
where \(e, f\) are minimal projections in \(V\) and \(\alpha, \beta, \lambda, \mu \in \mathbb{R}\). But (simple calculation) the Jordan subalgebra of \(V\) generated by \(e\) and \(f\) is linearly generated by \(e, f\) and \(ef + fe\), from which it follows that \(V = M_2(\mathbb{R})_{sa}\). By [10, Theorem 2], for example, we may therefore suppose that
\[
M = C(X) \otimes M_2(\mathbb{R})_{sa},
\]
where \(C(X)\) is the algebra of all continuous real-valued functions on a compact hyperstonean space \(X\).
We note that \( \mu(e) = \mu(f) \) whenever \( e, f \in P(M) \) satisfy
\[
(e \vee 1 - f) = 1 = (1 - e) \vee f.
\]
Indeed, because \( \mu \) is subadditive, the first equation of (1) implies that
\[
1 \leq \mu(e) + \mu(1 - f) = \mu(e) + 1 - \mu(f), \quad \text{so that} \quad \mu(f) \leq \mu(e).
\]
Let \( e, f \in P(M) \) with \( \|e - f\| < 1 \). Then \( e \wedge (1 - f) = 0 \), because
\[
e \wedge (1 - f) = (e - f)(e \wedge (1 - f)).
\]
Hence, \( (1 - e) \vee f = 1 \). Similarly, \( e \vee (1 - f) = 1 \). So \( \mu(e) = \mu(f) \), by (1). In particular, \( \mu \) is uniformly continuous on \( P(M) \).

Suppose now that \( e \) is any nontrivial projection \( (e \neq 0, 1) \) in \( N = 1 \otimes M_2(\mathbb{R})_{sa} \). Choose a projection \( f \neq 1 \) in \( N \) such that \( ef \neq 0 \). Then applying (1) to each of the pairs \( e, f \) and \( 1 - e, f \) in turn, we see that
\[
\mu(e) = \mu(f) = \mu(1 - e), \quad \text{so that} \quad \mu(e) = \frac{1}{2}.
\]
By the unicity of quasi-state extensions this means that \( \bar{\mu} \) can be none other than the (linear) trace on \( N \). Since for any projection \( z \) of \( C(X) \) there is a compact hyperstonean subspace \( Y \) of \( X \) with \( zC(X) = C(Y) \) and \( zM = C(Y) \otimes M_2(\mathbb{R})_{sa} \), it follows that \( \bar{\mu} \) is a (linear) trace on \( z \otimes M_2(\mathbb{R})_{sa} \). Thus, letting \( A \) be the Jordan subalgebra of \( M \) generated by all finite sums of the form
\[
\sum_{i=1}^{n} z_i \otimes M_2(\mathbb{R})_{sa} \quad \text{where the} \quad z_i \in P(C(X)) \quad \text{and} \quad \sum_{i=1}^{n} z_i = 1 \quad (n \in \mathbb{N}),
\]
it is immediate that \( \bar{\mu} \) is linear on \( A \) and that \( \bar{\mu}(xy^2x) = \bar{\mu}(yx^2y), \) for all \( x, y \in A \). But \( A \) is uniformly dense in \( M \) and also the set of projections of \( A \) is uniformly dense in \( P(M) \) (cf. [8, Lemma 8.3]). It follows directly from this, together with the above observed continuity of \( \bar{\mu} \) on \( P(M) \), that \( \bar{\mu} \) is linear on \( M \) and that \( \bar{\mu}(xy^2x) = \bar{\mu}(yx^2y), \) for all \( x, y \in M \). In other words, \( \bar{\mu} \) is a tracial state of \( M \).

(ii) Now let \( M \) be any JBW-algebra. By (i) and [3, 4] we may suppose that \( \bar{\mu} \) is linear on \( M \). Let \( e, f \in P(M) \). Since the JBW-subalgebra of \( M \) generated by \( e \) and \( f \) is of the form Type \( I_1 \oplus \text{Type } I_2 \), a further application of (i), above, together with part (iii) of [9, Theorem], implies that \( \bar{\mu}(e_0f) \geq 0 \). So, by spectral theory, \( \bar{\mu}(x_0y) \geq 0 \) for all \( x, y \in M_+ \), and the above-quoted result of [9] now implies that \( \bar{\mu} \) is a tracial state of \( M \).

**Corollary.** Let \( \mu \) be a subadditive normal semifinite measure on the projections \( P(M) \) of a JBW-algebra \( M \). Then \( \mu \) extends uniquely to a normal semifinite trace on \( M \).

**Proof.** The subadditivity of \( \mu \) means that \( S = \{e \in P(M) : \mu(e) < \infty \} \) is a sublattice of \( P(M) \). In particular, \( S \) is an increasing net which, by the semifiniteness of \( \mu \), converges strongly to \( 1 \). By the theorem and the fact that \( \mu \) is normal, for each \( e \in S \) the restriction of \( \mu \) to \( P(Ue(M)) \) extends uniquely to a finite normal tracial linear map \( \mu_{e} \) on \( Ue(M) \). The unicity of these extensions induces a linear mapping
\[
\mu_0 : \bigcup_{e \in S} Ue(M) \to \mathbb{R}, \quad \text{with} \quad \mu_0(x) = \mu_{e}(x) \text{ if } x \in Ue(M),
\]
extending $\mu$. Given $e, f \in S$, with $e \leq f$ and $x \in M_+$, we have, for $a = U_f(x)^{1/2},$

$$\mu_0(U_e(x)) = \mu_0(U_e(a^2)) = \mu_0(U_a(e)) \leq \mu_0(a^2) = \mu_0(U_f(x)).$$

Consequently, denoting $S$ by $\{e_a\}$ we may define a mapping

$$\overline{\mu} : M_+ \to [0, \infty) \text{ by } \overline{\mu}(x) = \lim \mu_0(U_{e_a}(x))$$

which extends $\mu_0$ on $M_+$ and satisfies, for $x, y \in M_+$ and $\lambda \geq 0$, (2)

$$\overline{\mu}(x + y) = \overline{\mu}(x) + \overline{\mu}(y) \text{ and } \overline{\mu}(\lambda x) = \lambda \overline{\mu}(x).$$

Now, with $p_{a\beta} = e_a \lor e_\beta$, $\overline{\mu}$ is a finite normal trace on $U_{p_{a\beta}}(M)$, so that for $x, y \in M$ we have that

$$\overline{\mu}(U_{e_a}U_xU_y(e_\beta)) = \overline{\mu}(U_{e_\beta}U_yU_x(e_a)).$$

Fixing $\beta$ and taking limits, this gives

$$\overline{\mu}(U_xU_y(e_\beta)) = \overline{\mu}(U_{e_\beta}U_y(x^2)) \to \overline{\mu}(U_y(x^2)).$$

To see that $\overline{\mu}$ must be normal, let $a_i \uparrow a$ in $M_+$. Then, respectively, $\overline{\mu}(U_{e_a}(a_i))$ and $\overline{\mu}(U_{e_a}(a))$ increase to $\overline{\mu}(a_i)$ and $\overline{\mu}(a)$ with respect to $\alpha$ while $\mu(U_{e_a}(a_i))$ increases to $\overline{\mu}(U_{e_a}(a))$ with respect to $i$. It follows that $\overline{\mu}(a_i) \to \overline{\mu}(a)$. So $\overline{\mu}$ is normal. The latter applied to (3) implies that $\overline{\mu}(U_x(y^2)) = \overline{\mu}(U_y(x^2))$, for all $x, y \in M$, which together with (2) shows $\overline{\mu}$ to be a normal trace on $M$. The semifiniteness of $\overline{\mu}$ is an immediate consequence of $U_{a^{1/2}}(e_a) \leq a$ and $\overline{\mu}(U_{a^{1/2}}(e_a)) = \overline{\mu}(U_{e_a}(a)) < \infty$, for all $a$. This completes the proof.

REFERENCES