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TRACES AND SUBADDITIVE MEASURES ON PROJECTIONS IN JBW-ALGEBRAS AND VON NEUMANN ALGEBRAS

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ABSTRACT. Let $P(M)$ be the projection lattice of an arbitrary JBW-algebra or von Neumann algebra M. It is shown that the tracial states of M correspond by extension precisely to the subadditive probability measures on $P(M)$. The analogous result for normal semifinite traces is also proved.

A *positive measure* on the projection lattice $P(M)$ of a JBW-algebra M is a function

$$\mu: P(M) \to [0, \infty] \text{ such that } \mu(e+f) = \mu(e) + \mu(f) \text{ whenever } ef = 0.$$

Such a measure is said to be a *probability* measure if $\mu(1) = 1$. A positive measure is said to be *normal* if $\mu(e_\alpha) \to \mu(e)$ whenever $e_\alpha \nearrow e$ and *semifinite* if for each projection e there is a net of projections $e_\alpha \nearrow e$ with $\mu(e_\alpha) < \infty$, for all α.

A *subadditive measure* on $P(M)$ is a positive measure such that

$$\mu(e \vee f) \leq \mu(e) + \mu(f), \quad \text{for all } e, f \in P(M).$$

By the generalisation of Gleason's Theorem [3, 4] each probability measure on $P(M)$ extends uniquely to a (linear) state on M whenever M has no Type I_2 direct summand. But this theorem is false for Type I_2 JBW-algebras.

The aim of this paper is to show that, for *every* JBW-algebra M, each subadditive probability measure on $P(M)$ extends uniquely to a *tracial* state on M. A corollary is that each subadditive normal semifinite measure on $P(M)$ extends to a normal semifinite trace on M.

In particular, this involves showing that, for subadditive measures, Gleason's Theorem is true for *all* JBW-algebras (including Type I_2 JBW-algebras).

We note that traces on JBW-algebras have been extensively studied in the literature [1, 2, 5, 7, 9] and have particular significance for the criterion of types. Our results have the particular virtue of providing a purely lattice- and measure-theoretic characterisation of traces on JBW-algebras.

Since the selfadjoint part of a von Neumann algebra is a JBW-algebra, we note that our results hold for all von Neumann algebras as well as for all JBW-algebras.

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The main reference for the theory of JBW-algebras is [6], to which the reader is referred for any unmentioned details. In order to facilitate the present discussion we recall that a JBW-algebra is a real Banach dual space and that a Jordan algebra (M, o) satisfies

$$\|x_0 y\| \leq \|x\| \, \|y\| \quad \text{and} \quad \|x\|^2 \leq \|x^2 + y^2\|.$$

A JW-algebra is a JBW-algebra that can be realized as a weakly closed Jordan algebra of selfadjoint operators on a Hilbert space. The Type I_2 JBW-algebras fall into this class and are l^∞-sums of algebras of the form $L^\infty(\Omega, \mu, V)$, where V is a spin factor and μ is a Radon measure on a locally compact Hausdorff space Ω [10].

A trace on a JBW-algebra M is a mapping $\tau: M^+ \to [0, \infty]$ for which

(i) $\tau(x+y) = \tau(x) + \tau(y)$, for all $x, y \in M^+$.
(ii) $\tau(\lambda x) = \lambda\tau(x)$, for all $x \in M^+$ and $\lambda \geq 0$.
(iii) $\tau(U_x(y^2)) = \tau(U_y(x^2))$, for all $x, y \in M$.

The notation used in (iii) is explained as follows. For each x in M, $U_x: M \to M$ is the positive map defined by $U_x(y) = 2x_0(x_0 y) - x_0^2 y$. (When M is a JW-algebra with product $x_0 y = \frac{1}{2}(xy + yx)$, this reduces to $U_x(y) = xyx$.) If $\tau(1) = 1$, then τ extends uniquely to a tracial state of M.

Let e and f be projections in a JBW-algebra M, and let τ be a trace on M. By [5, Lemma 2.3] together with [11, Corollary 8] there is a symmetry s in M with

$$U_s(e \vee f - f) = e - e \wedge f,$$

from which it follows easily that $\tau(e \vee f) \leq \tau(e) + \tau(f)$. Hence, every trace on M restricts to a subadditive measure on $P(M)$.

Theorem. *Let M be a JBW-algebra, and let μ be a subadditive probability measure on $P(M)$. Then μ extends uniquely to a tracial state on M.*

Proof. Being a probability measure, μ extends uniquely to a quasi-linear functional $\overline{\mu}$ on M [3, 4]. That is, $\overline{\mu}$ is linear (and continuous) on all abelian subalgebras of M.

(i) Let M be a Type I_2 JBW-algebra. Let $x, y \in M$. We have to show that
(a) $\overline{\mu}(x+y) = \overline{\mu}(x) + \overline{\mu}(y)$ and
(b) $\overline{\mu}(xy^2x) = \overline{\mu}(yx^2y)$.

Since the JBW-algebra, $M(x, y)$, of M generated by x and y is a direct sum of an abelian algebra and a Type I_2 algebra, no harm can can come from supposing that $M = M(x, y)$. In which case, let π be a factor representation of M. Then $\pi: M \to V$ where V is a spin factor and $\pi(M) = V$. So

$$\pi(x) = a1 + \beta e, \qquad \pi(y) = \lambda 1 + \mu f$$

where e, f are minimal projections in V and $\alpha, \beta, \lambda, \mu \in \mathbb{R}$. But (simple calculation) the Jordan subalgebra of V generated by e and f is linearly generated by e, f and $ef + fe$, from which it follows that $V = M_2(\mathbb{R})_{sa}$. By [10, Theorem 2], for example, we may therefore suppose that

$$M = C(X) \otimes M_2(\mathbb{R})_{sa},$$

where $C(X)$ is the algebra of all continuous real-valued functions on a compact hyperstonean space X.

We note that $\mu(e) = \mu(f)$ whenever $e, f \in P(M)$ satisfy

(1) $$e \vee (1-f) = 1 = (1-e) \vee f.$$

Indeed, because μ is subadditive, the first equation of (1) implies that

$$1 \leq \mu(e) + \mu(1-f) = \mu(e) + 1 - \mu(f), \quad \text{so that } \mu(f) \leq \mu(e).$$

Let $e, f \in P(M)$ with $\|e - f\| < 1$. Then $e \wedge (1-f) = 0$, because

$$e \wedge (1-f) = (e-f)(e \wedge (1-f)).$$

Hence, $(1-e) \vee f = 1$. Similarly, $e \vee (1-f) = 1$. So $\mu(e) = \mu(f)$, by (1). In particular, μ is uniformly continuous on $P(M)$.

Suppose now that e is any nontrivial projection ($\neq 0, 1$) in $N = 1 \otimes M_2(\mathbb{R})_{sa}$. Choose a projection $f \neq 1$ in N such that $ef \neq 0$. Then applying (1) to each of the pairs e, f and $1-e, f$ in turn, we see that

$$\mu(e) = \mu(f) = \mu(1-e), \quad \text{so that } \mu(e) = \tfrac{1}{2}.$$

By the unicity of quasi-state extensions this means that $\overline{\mu}$ can be none other than the (linear) trace on N. Since for any projection z of $C(X)$ there is a compact hyperstonean subspace Y of X with $zC(X) = C(Y)$ and $zM = C(Y) \otimes M_2(\mathbb{R})_{sa}$, it follows that $\overline{\mu}$ is a (linear) trace on $z \otimes M_2(\mathbb{R})_{sa}$. Thus, letting A be the Jordan subalgebra of M generated by all finite sums of the form

$$\sum_1^n z_i \otimes M_2(\mathbb{R})_{sa} \quad \text{where the } z_i \in P(C(X)) \text{ and } \sum_1^n z_i = 1 \ (n \in \mathbb{N}),$$

it is immediate that $\overline{\mu}$ is linear on A and that $\overline{\mu}(xy^2x) - \overline{\mu}(yx^2y)$, for all $x, y \in A$. But A is uniformly dense in M and also the set of projections of A is uniformly dense in $P(M)$ (cf. [8, Lemma 8.3]). It follows directly from this, together with the above observed continuity of $\overline{\mu}$ on $P(M)$, that $\overline{\mu}$ is linear on M and that $\overline{\mu}(xy^2x) = \overline{\mu}(yx^2y)$, for all $z, y \in M$. In other words, $\overline{\mu}$ is a tracial state of M.

(ii) Now let M be any JBW-algebra. By (i) and [3, 4] we may suppose that $\overline{\mu}$ is linear on M. Let $e, f \in P(M)$. Since the JBW-subalgebra of M generated by e and f is of the form Type $I_1 \oplus$ Type I_2, a further application of (i), above, together with part (iii) of [9, Theorem], implies that $\overline{\mu}(e_0 f) \geq 0$ So, by spectral theory, $\overline{\mu}(x_0 y) \geq 0$ for all $x, y \in M_+$, and the above-quoted result of [9] now implies that $\overline{\mu}$ is a tracial state of M.

Corollary. *Let μ be a subadditive normal semifinite measure on the projections $P(M)$ of a JBW-algebra M. Then μ extends uniquely to a normal semifinite trace on M.*

Proof. The subadditivity of μ means that $S = \{e \in P(M) : \mu(e) < \infty\}$ is a sublattice of $P(M)$. In particular, S is an increasing net which, by the semifiniteness of μ, converges strongly to 1. By the theorem and the fact that μ is normal, for each $e \in S$ the restriction of μ to $P(U_e(M))$ extends uniquely to a finite normal tracial linear map μ_e on $U_e(M)$. The unicity of these extensions induces a linear mapping

$$\mu_0 \colon \bigcup_{e \in S} U_e(M) \to \mathbb{R}, \quad \text{with } \mu_0(x) = \mu_e(x) \text{ if } x \in U_e(M),$$

extending μ. Given $e, f \in S$, with $e \leq f$ and $x \in M_+$, we have, for $a = U_f(x)^{1/2}$,

$$\mu_0(U_e(x)) = \mu_0(U_e(a^2)) = \mu_0(U_a(e)) \leq \mu_0(a^2) = \mu_0(U_f(x)).$$

Consequently, denoting S by $\{e_\alpha\}$ we may define a mapping

$$\overline{\mu}: M_+ \to [0, \infty] \quad \text{by} \quad \overline{\mu}(x) = \lim \mu_0(U_{e_\alpha}(x))$$

which extends μ_0 on M_+ and satisfies, for $x, y \in M_+$ and $\lambda \geq 0$,

$$(2) \qquad \overline{\mu}(x+y) = \overline{\mu}(x) + \overline{\mu}(y) \quad \text{and} \quad \overline{\mu}(\lambda x) = \lambda \overline{\mu}(x).$$

Now, with $p_{\alpha\beta} = e_\alpha \vee e_\beta$, $\overline{\mu}$ is a finite normal trace on $U_{p_{\alpha\beta}}(M)$, so that for $x, y \in M$ we have that

$$\overline{\mu}(U_{e_\alpha} U_x U_y(e_\beta)) = \overline{\mu}(U_{e_\beta} U_y U_x(e_\alpha)).$$

Fixing β and taking limits, this gives

$$(3) \qquad \overline{\mu}(U_x U_y(e_\beta)) = \overline{\mu}(U_{e_\beta} U_y(x^2)) \to \overline{\mu}(U_y(x^2)).$$

To see that $\overline{\mu}$ must be normal, let $a_i \uparrow a$ in M_+. Then, respectively, $\overline{\mu}(U_{e_\alpha}(a_i))$ and $\overline{\mu}(U_{e_\alpha}(a))$ increase to $\overline{\mu}(a_i)$ and $\overline{\mu}(a)$ with respect to α while $\overline{\mu}(U_{e_\alpha}(a_i))$ increases to $\overline{\mu}(U_{e_\alpha}(a))$ with respect to i. It follows that $\overline{\mu}(a_i) \to \overline{\mu}(a)$. So $\overline{\mu}$ is normal. The latter applied to (3) implies that $\overline{\mu}(U_x(y^2)) = \overline{\mu}(U_y(x^2))$, for all x, y in M, which together with (2) shows $\overline{\mu}$ to be a normal trace on M. The semifiniteness of $\overline{\mu}$ is an immediate consequence of $U_{a^{1/2}}(e^\alpha) \leq a$ and $\overline{\mu}(U_{a^{1/2}}(e_\alpha)) = \overline{\mu}(U_{e_\alpha}(a)) < \infty$, for all a. This completes the proof.

REFERENCES

1. S. A. Ajupov, *Extension of traces and type criterions for Jordan algebras of self-adjoint operators*, Math. Z. **181** (1982), 253–268.
2. S. A. Ajupov and R. Z. Abdullev, *The Radon Nikodym Theorem for weights on JBW-algebras*, Math. Z. **188** (1985), 475–484.
3. L. J. Bunce and J. D. M. Wright, *Quantum measures and states on Jordan algebras*, Comm. Math. Phys. **98** (1985), 187–202.
4. ——, *Continuity and linear extensions of quantum measures on Jordan algebras*, Math. Scand. **64** (1989), 300–306.
5. U. Haagerup and H. Hanche-Olsen, *Tomita-Takesaki Theory for Jordan algebras*, J. Operator Theory **11** (1984), 343–364.
6. H. Hanche-Olsen and E. Størmer, *Jordan operator algebras*, Pitman, New York, 1984.
7. B. Iochum, *Cônes autopolaires et algèbres dde Jordan*, Lecture Notes in Math., vol. 1049, Springer-Verlag, New York, 1984.
8. S. Maeda, *Probability measures on projections in von Neumann algebras*, Rev. Math. Phys. **1** (1990), 235–290.
9. G. K. Pedersen and E. Størmer, *Traces on Jordan algebras*, Canad. J. Math. **34** (1982), 370–373.
10. P. J. Stacey, *Type I_2 JBW-algebras*, Quart. J. Math. Oxford **33** (1982), 115–127.
11. D. Topping, *Jordan Algebras of self-adjoint operators*, Mem. Amer. Math. Soc., vol. 53, Amer. Math. Soc., Providence, RI, 1965.

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