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TOPOLOGY OF FACTORED ARRANGEMENTS OF LINES

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ABSTRACT. A real arrangement of affine lines is a finite family $\mathscr{A}$ of lines in $\mathbf{R}^2$. A real arrangement $\mathscr{A}$ of lines is said to be factored if there exists a partition $\Pi = (\Pi_1, \Pi_2)$ of $\mathscr{A}$ into two disjoint subsets such that the Orlik-Solomon algebra of $\mathscr{A}$ factors according to this partition. We prove that the complement of the complexification of a factored real arrangement of lines is a $K(\pi, 1)$ space.

1. INTRODUCTION

Let $\mathbf{K}$ be a field, and let V be a vector space over $\mathbf{K}$. An *arrangement of (affine) hyperplanes* in V is a finite family $\mathscr{A}$ of (affine) hyperplanes of V. An *arrangement of (affine) lines* is an arrangement of hyperplanes in a 2-dimensional vector space $V = \mathbf{K}^2$. An arrangement $\mathscr{A}$ of hyperplanes is said to be *real* (resp. *complex*) if $\mathbf{K} = \mathbf{R}$ is the field of real numbers (resp. if $\mathbf{K} = \mathbf{C}$ is the field of complex numbers). The *complexification* of a hyperplane H of $\mathbf{R}^l$ is the hyperplane $H_\mathbf{C}$ of $\mathbf{C}^l$ having the same equation as H. The *complexification* of a real arrangement $\mathscr{A}$ of hyperplanes in $\mathbf{R}^l$ is the arrangement $\mathscr{A}_\mathbf{C} = \{H_\mathbf{C} \mid H \in \mathscr{A}\}$ in $\mathbf{C}^l$.

Let $\mathscr{A}$ be a complex arrangement of hyperplanes in $V = \mathbf{C}^l$. The *complement* of $\mathscr{A}$ is the connected submanifold

$$M(\mathscr{A}) = V - \left(\bigcup_{H \in \mathscr{A}} H \right)$$

of V. We say that $\mathscr{A}$ is a $K(\pi, 1)$ *arrangement* if $M(\mathscr{A})$ is a $K(\pi, 1)$ space. We say that a real arrangement $\mathscr{A}$ of hyperplanes is a $K(\pi, 1)$ *arrangement* if its complexification $\mathscr{A}_\mathbf{C}$ is a $K(\pi, 1)$ arrangement. Yet, only two classes of real $K(\pi, 1)$ arrangements of hyperplanes are known. These are the simplicial arrangements (see [De]) and supersolvable arrangements (see [Te1]). Other examples of real $K(\pi, 1)$ arrangements appear in [Fa] and in [JS].

Our aim in this paper is to produce a new class of real $K(\pi, 1)$ arrangements: "factored arrangements of lines". This class contains supersolvable arrangements of lines (see [Ja]).

We refer to [FR] for a good exposition on $K(\pi, 1)$ arrangements.

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Let $\mathscr{A}$ be an arrangement of hyperplanes. The *intersection poset* of $\mathscr{A}$ is the ranked poset $\mathscr{L}(\mathscr{A})$ consisting of all nonempty intersections of elements of $\mathscr{A}$ ordered by reverse inclusion. $V = \bigcap_{H \in \varnothing} H$ is assumed to be the smallest element of $\mathscr{L}(\mathscr{A})$. For $X \in \mathscr{L}(\mathscr{A})$, we set

$$\mathscr{A}_X = \{H \in \mathscr{A} \mid H \supseteq X\}.$$

A partition $\Pi = (\Pi_1, \dots, \Pi_l)$ of $\mathscr{A}$ into l disjoint nonempty subsets is called *independent* if, for any choice of hyperplanes $H_i \in \Pi_i$ $(i = 1, \dots, l)$, the subspace $H_1 \cap \cdots \cap H_l$ is nonempty and its rank is l in $\mathscr{L}(\mathscr{A})$. If $X \in \mathscr{L}(\mathscr{A})$, then Π *induces* a partition Π_X of $\mathscr{A}_X$ whose blocks are the nonempty subsets $\Pi_i \cap \mathscr{A}_X$. A partition $\Pi = (\Pi_1, \dots, \Pi_l)$ of $\mathscr{A}$ is a *factorization* (or a *nice partition*) if

(1) Π is independent;

(2) if $X \in \mathscr{L}(\mathscr{A}) - \{V\}$, then Π_X has at least a block which is a singleton.

If $\mathscr{A}$ is an arrangement of lines, then any factorization of $\mathscr{A}$ has to be a partition $\Pi = (\Pi_1, \Pi_2)$ of $\mathscr{A}$ into two disjoint subsets (see [Te2]). We say that an arrangement $\mathscr{A}$ of hyperplanes is *factored* if $\mathscr{A}$ has a factorization.

Factored arrangements have been introduced and investigated by Falk, Jambu, and Terao [FJ, Te2]. One of the main results concerning these arrangements is the following theorem due to Terao [Te2].

The homogeneous component $A^1(\mathscr{A})$ of the Orlik-Solomon algebra $A(\mathscr{A})$ of an arrangement $\mathscr{A}$ of hyperplanes can be viewed as a free $\mathbf{Z}$-module spanned by the hyperplanes of $\mathscr{A}$ (see [OS]). For $\mathscr{B} \subseteq \mathscr{A}$, we denote by $B(\mathscr{B})$ the submodule of $A^1(\mathscr{A})$ spanned by the elements of $\mathscr{B}$.

Theorem 1 (Terao [Te2]). *Let $\mathscr{A}$ be an arrangement of hyperplanes. Let $\Pi = (\Pi_1, \dots, \Pi_l)$ be a partition of $\mathscr{A}$. The Orlik-Solomon algebra of $\mathscr{A}$, viewed as a graded $\mathbf{Z}$-module, factors as*

$$A(\mathscr{A}) = (\mathbf{Z} \oplus B(\Pi_1)) \otimes \cdots \otimes (\mathbf{Z} \oplus B(\Pi_l))$$

if and only if Π is a factorization.

Our goal in this paper is to prove the following theorem.

Theorem 2. *If $\mathscr{A}$ is a factored real arrangement of lines, then $\mathscr{A}$ is a $K(\pi, 1)$ arrangement.*

Example. Consider the arrangement $\mathscr{A}$ shown in Figure 1. Set $\Pi_1 = \{l_1, l_2, l_3, l_4\}$ and $\Pi_2 = \{l_5, l_6, l_7, l_8\}$. Then $\Pi = (\Pi_1, \Pi_2)$ is a factorization of $\mathscr{A}$. Note that this arrangement is neither simplicial nor supersolvable.

A direct consequence of Theorem 2 is the following corollary. Recall that an arrangement $\mathscr{A}$ of hyperplanes is said to be *central* if $\bigcap_{H \in \mathscr{A}} H \neq \varnothing$.

Corollary. *Let $\mathscr{A}$ be a real and central arrangement of hyperplanes. Assume that the rank of $\mathscr{L}(\mathscr{A})$ is 3. If $\mathscr{A}$ is a factored arrangement, then $\mathscr{A}$ is a $K(\pi, 1)$ arrangement.*

Proof. Let $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ be a factorization of $\mathscr{A}$. One may assume that $\mathscr{A}$ is an arrangement in $\mathbf{R}^3$, that $\bigcap_{H \in \mathscr{A}} H = \{0\}$, and that Π_3 is a singleton $\{H_0\}$. Let K_0 be an (affine) hyperplane of $\mathbf{R}^3$ parallel to H_0 and different

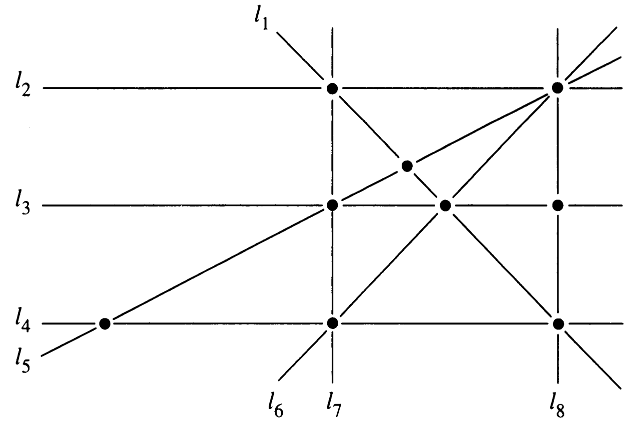

FIGURE 1

from H_0. Set

$$\widetilde{\mathscr{A}} = \{H \cap K_0 \mid H \in \mathscr{A} - \{H_0\}\},$$
$$\widetilde{\Pi}_1 = \{H \cap K_0 \mid H \in \Pi_1\},$$
$$\widetilde{\Pi}_2 = \{H \cap K_0 \mid H \in \Pi_2\}.$$

Then $\widetilde{\mathscr{A}}$ is a real arrangement of lines in K_0, the partition $\widetilde{\Pi} = (\widetilde{\Pi}_1, \widetilde{\Pi}_2)$ is a factorization of $\widetilde{\mathscr{A}}$, and $M(\mathscr{A}_{\mathbf{C}})$ is homeomorphic to $\mathbf{C}^* \times M(\widetilde{\mathscr{A}_{\mathbf{C}}})$ (see [OT, Proposition 5.1.1]). So, $M(\mathscr{A}_{\mathbf{C}})$ is a $K(\pi, 1)$ space since, by Theorem 2, $M(\widetilde{\mathscr{A}_{\mathbf{C}}})$ is a $K(\pi, 1)$ space. □

The proof of Theorem 2 is a direct application of Falk's weight test for a real arrangement of lines to be $K(\pi, 1)$ (see [Fa]).

Section 2 is divided into two subsections. In §2.1 we state Falk's weight test (Theorem 3). In §2.2 we prove Theorem 2.

2. PROOF OF THEOREM 2

Throughout this section $\mathscr{A}$ is assumed to be an arrangement of affine lines in $V = \mathbf{R}^2$.

2.1. **Falk's weight test for $K(\pi, 1)$ arrangements.** The lines of $\mathscr{A}$ subdivide V into *facets.* The *support* $|f|$ of a facet f is the smallest affine subspace of V containing f. Every facet is open in its support. We denote by $\overline{f}$ the closure of f in V. There is a partial order on the set of facets defined by $f \leq g$ if $f \subseteq \overline{g}$. 0-dimensional facets are called *vertices,* 1-dimensional facets are called *edges,* and 2-dimensional facets are called *faces.*

Let $\Gamma(\mathscr{A})$ denote the planar 2-complex consisting of the bounded facets. We denote by $\Gamma^{(i)}(A)$ its i-skeleton $(i = 0, 1, 2)$. A *corner* of $\Gamma(\mathscr{A})$ is a chain $(v < f)$ with $v \in \Gamma^{(0)}(\mathscr{A})$ and $f \in \Gamma^{(2)}(\mathscr{A})$. We denote by $\mathrm{Corn}(\mathscr{A})$ the set of corners. A *system of weights* on $\Gamma(\mathscr{A})$ is a function $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+ = [0, +\infty[$.

Let $v \in \Gamma^{(0)}(\mathscr{A})$. The *link graph* of $\Gamma(\mathscr{A})$ at v is the graph Λ_v defined as follows.

(1) The vertices of Λ_v are the chains $(v < e)$ with $e \in \Gamma^{(1)}(\mathscr{A})$.

(2) The edges of Λ_v are the chains (or corners) $(v < f)$ with $f \in \Gamma^{(2)}(\mathscr{A})$.

(3) An edge $(v < f)$ is incident with a vertex $(v < e)$ if $v < e < f$.

Let $\gamma = ((v < e_0), (v < e_1), \dots, (v < e_n))$ be a path in Λ_v. For $j = 1, \dots, n$, let $(v < f_j)$ be the edge of Λ_v incident with $(v < e_{j-1})$ and $(v < e_j)$. For a given system of weights $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$, we define the *weight* of γ to be

$$\Omega(\gamma) = \sum_{j=1}^{n} \Omega(v < f_j).$$

A path $\gamma = ((v < e_0), (v < e_1), \dots, (v < e_n))$ is a *circuit* if $e_0 = e_n$. Let $\mathscr{A}_v$ denote the set of lines of $\mathscr{A}$ which contain v. A circuit $\gamma = ((v < e_0), (v < e_1), \dots, (v < e_n))$ is *full* if, for every $l \in \mathscr{A}_v$, there exist at least two distinct indices $1 \le j < k \le n$ such that $|e_j| = |e_k| = l$. A system of weights $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$ is said to be *$\mathscr{A}$-admissible* if, for every $v \in \Gamma^{(0)}(\mathscr{A})$ and every full circuit γ of Λ_v, we have $\Omega(\gamma) \ge 2\pi$.

Let $f \in \Gamma^{(2)}(\mathscr{A})$. We denote by $d(f)$ the number of vertices $v \in \Gamma^{(0)}(\mathscr{A})$ such that $v < f$. It is also the number of edges $e \in \Gamma^{(1)}(\mathscr{A})$ such that $e < f$. For a given system of weights $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$, we define the *weight* of f to be

$$\Omega(f) = \sum_{v<f} \Omega(v < f).$$

A system of weights $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$ is said to be *aspherical* if, for every $f \in \Gamma^{(2)}(\mathscr{A})$, we have $\Omega(f) \le (d(f) - 2)\pi$.

Theorem 3 (Falk [Fa]). *If $\Gamma(\mathscr{A})$ admits a system of weights which is $\mathscr{A}$-admissible and aspherical, then $\mathscr{A}$ is a $K(\pi, 1)$ arrangement.*

Remark. There is a similar criterium given in [JS] for a real arrangement of lines to be $K(\pi, 1)$.

2.2. **Proof of Theorem 2.** Let $\Pi = (\Pi_1, \Pi_2)$ be a factorization of $\mathscr{A}$. Let $(v < f)$ be a corner of $\Gamma(\mathscr{A})$. Let e_1 and e_2 be the two edges of $\Gamma(\mathscr{A})$ such that $v < e_i < f$ $(i = 1, 2)$. We say that $(v < f)$ is *coloured* if, up to some permutation, $|e_1| \in \Pi_1$ and $|e_2| \in \Pi_2$. We consider the system of weights defined by

$$\Omega(v < f) = \begin{cases} \dfrac{\pi}{2} & \text{if } (v < f) \text{ is coloured}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $v \in \Gamma^{(0)}(\mathscr{A})$. Let $\gamma = ((v < e_0), (v < e_1), \dots, (v < e_n))$ be a full circuit of Λ_v. For $j = 1, \dots, n$, let $(v < f_j)$ be the edge of Λ_v incident with $(v < e_{j-1})$ and $(v < e_j)$. By definition of a factorization, we may assume that $\mathscr{A}_v \cap \Pi_1$ is a singleton $\{l_0\}$. By definition of a full circuit, there exist two indices $1 \le j < k \le n$ such that $|e_j| = |e_k| = l_0$. Obviously, $j \ne k - 1$ and $k \ne j - 1$ (we assume that $j - 1 = n$ if $j = 1$), and $(v < f_{j-1}), (v < f_j), (v < f_{k-1})$, and $(v < f_k)$ are coloured corners. Thus,

$$\Omega(\gamma) \ge \Omega(v < f_{j-1}) + \Omega(v < f_j) + \Omega(v < f_{k-1}) + \Omega(v < f_k) = 2\pi.$$

This shows that $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$ is $\mathscr{A}$-admissible.

Let $f \in \Gamma^{(2)}(\mathscr{A})$. If f is a triangle (i.e., $d(f) = 3$), then there exist at most two vertices $v_1, v_2 \in \Gamma^{(0)}(\mathscr{A})$ such that $v_i < f$ and the corner $(v_i < f)$ is coloured (for $i = 1, 2$). Thus,

$$\Omega(f) \leq 2 \cdot \frac{\pi}{2} = (d(f) - 2)\pi.$$

If $d(f) \geq 4$, then

$$\Omega(f) \leq d(f)\frac{\pi}{2} \leq (d(f) - 2)\pi.$$

This shows that $\Omega\colon \mathrm{Corn}(\mathscr{A}) \to \mathbf{R}^+$ is aspherical. □

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