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REAL ANALYTIC SUBMANIFOLDS UNDER UNIMODULAR TRANSFORMATIONS

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ABSTRACT. We study the normal form of totally real and real analytic submanifolds in $\mathbb{C}^n$ under holomorphic unimodular transformations. We also consider the unimodular normal form of real surfaces in $\mathbb{C}^2$ near an elliptic complex tangent with the non-vanishing Bishop invariant or near a non-exceptional hyperbolic complex tangent.

1. INTRODUCTION

In this paper, we study the normal forms of n-dimensional real analytic submanifolds in $\mathbb{C}^n (n \geq 2)$ under unimodular transformations. By a unimodular transformation we mean a local biholomorphic mapping which preserves the n-form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ on $\mathbb{C}^n$. For a totally real and real analytic submanifold $M^k \subset \mathbb{C}^n$ of dimension k, we will prove that it is locally equivalent to the standard $\mathbb{R}^k \subset \mathbb{C}^n$ under unimodular transformations if $k < n$. When $k = n$, there exists an invariant function attached to $M = M^n$. To describe this, we fix $p \in M$ and put

$$\omega_M = \Omega|_M = e^{i\theta} V, \qquad \theta(p) \in [0, \pi), \tag{1.1}$$

where θ is a real analytic function and V is a non-vanishing real n-form on M. We will see that M is locally equivalent to the standard $\mathbb{R}^k \subset \mathbb{C}^n$ under unimodular transformations defined near p if and only if θ vanishes. In section 2, we will use Vey's Morse lemma [6] to prove the following:

Theorem 1.1. *Suppose that $M \subset \mathbb{C}^n (n \geq 2)$ is a totally real and real analytic n-dimensional submanifold containing p. Let θ be given by* (1.1). *If p is a non-degenerate critical point of θ, then there is a unimodular transformaton ϕ defined near p with $\phi(p) = 0$ such that*

$$\tilde{M} = \phi(M) : \operatorname{Im} z_1 = \operatorname{Im} z_{n-1} = \operatorname{Im}\{z_n a(z)\} = 0,$$

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in which $a(z)$ *is given by*

$$(1.2)\qquad \frac{\partial}{\partial z_n}\{z_n a(z)\} = \exp\{-i\sum_{k=0}^{\infty} c_k h(z_1, \dots, z_{n-1}, z_n a(z))^k\},$$

$$h(z) = \sum_{j=1}^{s} z_j^2 - \sum_{j=s+1}^{n} z_j^2, \quad 1 \le s \le n.$$

Moreover, s *and coefficients* c_j $(0 \le j < \infty)$ *give the full set of unimodular invariants of* M.

We now consider a real analytic surface in $\mathbb{C}^2$ with a complex tangent at 0, which is given by

$$(1.3)\qquad M : z_2 = z_1\overline{z}_1 + \gamma z_1^2 + \overline{\gamma}\,\overline{z}_1^2 + H(z_1, \overline{z}_1), \qquad \gamma \in \mathbb{C},$$

in which the power series $H(z_1, \overline{z}_1)$ begins with the third-order terms, γ is a unimodular invariant, and $|\gamma|$ is the Bishop invariant [2]. The complex tangent of M at 0 is said to be *elliptic*, *parabolic*, or *hyperbolic* according to $0 \le |\gamma| < 1/2$, $|\gamma| = 1/2$, or $1/2 < |\gamma| < \infty$, respectively. Let λ be one of the roots of $|\gamma|\lambda^2 + \lambda + |\gamma| = 0$. γ is said to be *non-exceptional* if λ is not a root of unity. A *formal* unimodular transformation of $\mathbb{C}^n$ is defined by $\phi(z) = (\phi_1(z), \dots, \phi_n(z))$, where $\phi_j(z)$ are given by formal power series in z without the constant term, and $\phi(z)$ preserves $\mathbf{\Omega}$. We have

Theorem 1.2. *Let* $M \subset \mathbb{C}^2$ *be a formal surface given by* (1.3). *Assume that its Bishop invariant* γ *satisfies* $0 < |\gamma| < 1/2$ *or is non-exceptional if* $1/2 < |\gamma| < \infty$. *Then there exists a unique formal unimodular transformation* ϕ *such that* $\phi(M)$ *is given by*

$$(1.4)\qquad \begin{cases} x_2 = \zeta\overline{\zeta} + (1 + \rho x_2^s)(\gamma\zeta + \overline{\gamma}\,\overline{\zeta}^2), \\ y_2 = 0, \qquad \zeta = \alpha(z_1, x_2) z_1, \end{cases}$$

where $\rho \in \mathbb{R} \setminus \{0\}$ *and* s *is a positive integer, or* $\rho = 0$ *with* $s = \infty$. *Furthermore,* $\alpha(z_1, z_2)$ *satisfies the normalizing conditions*:

(i) $\alpha(0) = 1$ *for* $\rho \neq 0$; *or*
(ii) $\alpha(0, z_2) = e^{i\theta(z_2)}$, $\theta(0) = 0$, *and* $\overline{\theta}(z_2) = \theta(z_2)$ *if* $\rho = 0$.

The ingredients for deriving (1.4) are the Moser-Webster normal form and the group of formal automorphisms of a real analytic surface given in [5]. By considering the action of the group of automorphisms on a suitable functional space, we are able to find our normal form and to show that (1.4) is realized by a convergent unimodular transformation if and only if its Moser-Webster normal form can be realized by a convergent biholomorphic mapping. In particular, this implies that the unimodular transformation converges when $0 < |\gamma| < 1/2$. It also shows that divergence occurs when $1/2 < |\gamma| < \infty$. One notices that $|\gamma|$, s, and the sign of ρ are the full set of invariants given in [5].

2. Totally real submanifolds

Throughout the discussion of normal forms, mappings and transformations are always defined in small open sets which are not specified. In particular,

$f:(X, x_0) \to (Y, y_0)$ denotes a mapping which is defined near x_0 and $f(x_0) = y_0$.

We now consider the normal form of a totally real and real analytic submanifold $M^k \subset \mathbb{C}^n (n \geq 2)$. For $k < n$, we want to show that M^k is equivalent to the standard $\mathbb{R}^k$ by unimodular transformations. For the proof, let $\xi = \varphi(z)$ be local holomorphic coordinates such that $\varphi(0) = 0$ and $\varphi(M) = \mathbb{R}^k$:

$$\operatorname{Im} \xi_1 = \cdots = \operatorname{Im} \xi_k = \xi_{k+1} = \cdots = \xi_n = 0.$$

One can decompose uniquely $\varphi = \sigma \circ \tau$, where $\xi = \sigma(w)$ fixes pointwise the hyperplane $w_n = 0$, and $w = \tau(z)$ is a unimodular transformation. To see this, we put

$$w = \sigma^{-1}(\xi) = (\xi', \xi_n a(\xi', \xi_n)), \quad \xi' = (\xi_1, \dots, \xi_{n-1}).$$

Denote by D the Jacobian matrix for a given mapping. We have

$$\det(D\sigma^{-1}) = \det(D(\tau \circ \varphi^{-1})), \qquad \det(D\tau) \equiv 1.$$

Hence

$$\frac{\partial}{\partial \xi_n} \xi_n a(\xi', \xi_n) = \det(D\varphi^{-1}(\xi', \xi_n)). \tag{2.1}$$

Clearly, the solution to (2.1) exists uniquely. We now obtain that $\tau(M) = \sigma^{-1}(\mathbb{R}^k)$:

$$\begin{cases} w_n = r(x'), & x' = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_k) \\ \operatorname{Im} w_1 = \cdots = \operatorname{Im} w_k = 0, & w_{k+1} = \cdots = w_{n-1} = 0, \end{cases}$$

where $r(x')$ converges near $0 \in \mathbb{R}^k$. Let

$$\tau_0(w) = (w_1, \dots, w_{n-1}, w_n - r(w_1, \dots, w_k)).$$

Then $\psi = \tau_0 \circ \tau$ is unimodular, and locally $\psi(M) = \mathbb{R}^k$.

The above argument also shows that a totally real and real analytic submanifold $M^n \subset \mathbb{C}^n$ may be locally transformed into $\mathbb{R}^{n-1} \times \mathbb{C}$ under unimodular transformations. However, there exists a functional module of obstructions to straighten M^n to be $\mathbb{R}^n$.

Let $M^n \subset \mathbb{C}^n$ be a totally real and real analytic submanifold containing p. Since M is totally real, we have $\omega_M \neq 0$. To describe the intrinsic property of ω_M, we consider an arbitrary real analytic parameterization $\varphi : (\mathbb{R}^n, 0) \to (M, p)$ and let $\omega_\varphi = \varphi^* \omega_M$. Obviously, ω_φ is a nowhere vanishing real analytic n-form. Conversely, for a nowhere vanishing $\mathbb{C}$-valued real analytic n-form ω defined near 0, there exists a totally real and real analytic submanifold $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{C}^n, 0)$ which realizes ω. To see this, we write $\omega = a(x) dx_1 \wedge \cdots \wedge dx_n$. Then there exists a $\mathbb{C}$-valued real analytic function $b(x)$ given by the equation

$$\frac{\partial b(x)}{\partial x_n} = a(x), \qquad b(x) \equiv 0 \quad \text{for } x_n = 0.$$

We define $\varphi(x) = (x_1, \dots, x_{n-1}, b(x))$. Thus $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{C}^n, 0)$ is a realization of ω.

Lemma 2.1. *Suppose that $M, \widetilde{M} \subset \mathbb{C}^n (n \geq 2)$ are two totally real and real analytic submanifolds containing 0. Let $\varphi, \widetilde{\varphi} : (\mathbb{R}^n, 0) \to (\mathbb{C}^n, 0)$ be real* *analytic parameterizations for* M *and* $\widetilde{M}$*, respectively. Then* M *and* $\widetilde{M}$ *are equivalent under unimodular transformations with* 0 *fixed if and only if there exists a real analytic mapping* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *such that* $f^*\omega_{\widetilde{\varphi}} = \omega_\varphi$.

Proof. We first assume that there exists a unimodular transformation $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $F(M) = \widetilde{M}$. Let $f = \widetilde{\varphi}^{-1} \circ F \circ \varphi$. Then $f^*\omega_{\widetilde{\varphi}} = \omega_\varphi$. Conversely, given a real analytic mapping $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $f^*\omega_{\widetilde{\varphi}} = \omega_\varphi$, we complexify $\widetilde{\varphi} \circ f \circ \varphi^{-1}$ to be $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Clearly, $F^*\Omega|_M = \Omega|_M$. Since both are holomorphic n-forms and M is real analytic, this implies that $F^*\Omega = \Omega$. □

To classify $\mathbb{C}$-valued real analytic n-forms on $\mathbb{R}^n$, we write $\omega = \rho\mu dx_1 \wedge \cdots \wedge dx_n$ where ρ, μ are real analytic, $\rho > 0$, and $|\mu| = 1$. By a change of real analytic coordinates, one can achieve $\rho \equiv 1$. One may also replace $\mu(0)$ by $-\mu(0)$ via reversing the orientation of $\mathbb{R}^n$. Therefore, we may restrict ourselves to the set of n-forms which have the form

$$\omega = e^{i\theta(x)} dx_1 \wedge \cdots \wedge dx_n, \qquad \theta(0) \in [0, \pi), \tag{2.2}$$

where $\theta(x)$ is real analytic. The group of transformations acting on n-forms (2.2) is the set of real analytic mappings which fix the origin and preserve the standard volume form on $\mathbb{R}^n$.

As a consequence of Lemma 2.1, we have

Corollary 2.2. *Suppose that* $M \subset \mathbb{C}^n$ $(n \geq 2)$ *is a totally real and real analytic submanifold of dimension* n *containing* p. *Let* θ *be defined by* (1.1). *Then* M *is unimodularly equivalent to a linear subspace if and only if* θ *is constant near* p. *In the general position, i.e.,* $d\theta(p) \neq 0$, M *can be transformed into*

$$y_1 = \cdots = y_{n-1} = \operatorname{Im} e^{-i(\theta(0)+x_1)} z_n = 0$$

through a unimodular transformation ϕ *with* $\phi(p) = 0$.

We now understand that classifying totally real manifolds under unimodular transformation is the same as classifying real analytic functions under volume-preserving mappings.

Proof of Theorem 1.1. Choose a real analytic parameterization $\phi : (\mathbb{R}^n, 0) \to (M^n, p)$ such that $\phi^*\omega_M$ is given by (2.2). By Vey's Morse lemma [6], there exists an analytic mapping $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ which preserves the standard volume form on $\mathbb{R}^n$ and transforms $\theta(x)$ into

$$\tilde{\theta}(w) = \theta(f^{-1}(w)) = \sum_{k=0}^{\infty} c_k h(w)^k, \tag{2.3}$$

where $h(w) = \sum_{j=1}^{s} w_j^2 - \sum_{j=s+1}^{n} w_j^2 \ (s \geq 1)$. Furthermore, $\{s, c_0, c_1, \ldots\}$ is the full set of unimodular invariants for the function θ. Hence for a volume-preserving transformation, we may assume that (2.2) has the form

$$\tilde{\omega} = e^{i\tilde{\theta}(w)} dw_1 \wedge \cdots \wedge dw_n. \tag{2.4}$$

We need to construct a mapping $\tilde{\varphi} : (\mathbb{R}^n, 0) \to (\tilde{M}, 0) \subset (\mathbb{C}^n, 0)$ such that $\tilde{\varphi}$ realizes the n-form (2.4), while $\tilde{M}$ has the normal form as stated in the theorem. To this end, let $z = \tilde{\varphi}(w) = (w', w_n b(w', w_n))$ or

$$w = \tilde{\varphi}^{-1}(z', z_n) = (z', z_n a(z', z_n)). \tag{2.5}$$

Set $\tilde{\varphi}^*\Omega = \tilde{\omega}$. From (2.4) and (2.5), we have

$$\det(D\tilde{\varphi}^{-1}(z)) = \exp\{-i\,\tilde{\theta}(\tilde{\varphi}^{-1}(z))\},$$

which, in view of (2.3), is exactly the equation (1.2). Therefore, by (2.5), the solution to (1.2) defines a mapping $\tilde{\varphi}$ such that $\tilde{\varphi}(\mathbb{R}^n)$ is the normal form for the given M. One notes that the convergence of the solution follows easily from (2.1). Since $\{p, c_0, c_1, \dots\}$ is the full set of unimodular invarians of θ, it also provides the full set of unimodular invariants for M by Lemma 2.1. This proves the theorem. □

To discuss the global invariant, we assume that N is an n-dimensional complex manifold with a non-vanishing holomorphic n-form Ω. Let $f: M \to N$ be a totally real immersion or embedding. Assume that M is orientable and is given a volume form ω. Write $f^*\Omega = \rho\mu\omega$, where ρ and μ are smooth functions with $\rho > 0$, $|\mu| = 1$. Alternatively, μ can be defined as follows. Choose a unimodular frame $v_1, \dots, v_n$ for T_xM. Let $e_1, \dots, e_n$ be a unimodular frame for $T^{(1,0)}_{f(x)}N$. Then $df_x(v_1, \dots, v_n) = (e_1, \dots, e_n)A(x)$, where $A(x) \in G(n, \mathbb{C})$. It is easy to see that $\rho(x)\mu(x) = \det(A(x))$. Hence $\mu(x) = \det(A(x))/|\det(A(x))|$. Denote by ϑ_f the pull-back of the generator $1 \in \mathbf{H}^1(S^1, \mathbb{Z}) \simeq \mathbb{Z}$ by $\mu: M \to S^1$. Clearly, a regular homotopy of totally real immersions induces a homotopy for the invariant function μ. This implies that, for $f: M \to N$, $\vartheta_f \in H^1(M, \mathbb{Z})$ is invariant under regular homotopy of totally real immersions.

We notice a standard fact due to Moser [4] that when M is compact, there exists a 1-parameter family of automorphisms ϕ_t of M such that

$$\phi_0 = \mathrm{Id}, \qquad \phi_1^*(\rho\omega) = c\omega,$$

in which $c = \int_M \rho\omega / \int_M \omega$. Therefore, for a compact and orientable manifold M of volume form ω, a totally real immersion $f: M \to N$ is regular homotopic to a totally real immersion $\widetilde{f}: M \to N$ which satisfies $\widetilde{f}^*\Omega = c\mu\omega$, where c is constant. One can see that c is a global unimodular invariant.

We should mention here that the above ϑ_f for totally real immersion $f: M \to N$ has appeared in [3], where F. Forstnerič defined an index homomorphism from $H_1(M, \mathbb{Z})$ into $\mathbb{Z}$ which is analogous to a definition of Maslov index for Lagrangian manifold given by V. I. Arnol'd [1]. Here we consider totally real immersions in an ambient complex manifold N of which the canonical line bundle is trivial. In fact, when the ambient space N is compact or simply connected, the cohomology class $\boldsymbol{\vartheta}$ is independent of the choices of holomorphic volume forms on N.

3. Normal forms of Moser and Webster

We will recall the Moser-Webster normal forms of real surfaces under biholomorphic transformations, which will be used in the next section.

Let $M \subset \mathbb{C}^2$ be a real analytic submanifold with a complex tangent at $0 \in M$. For a linear unimodular change of coordinates, we may assume the z_1-axis is tangent to M at 0. Then M is given by $z_2 = az_1^2 + bz_1\overline{z}_1 + c\overline{z}_1^2 + O(|z_1|^3)$. We make a non-degeneracy assumption that $b \neq 0$. Applying the unimodular transformation

$$(z_1, z_2) \mapsto (b|b|^{-\frac{2}{3}} z_1, b^{-1}|b|^{\frac{2}{3}} z_2),$$

we achieve $b = 1$. Hence $M : z_2 = \tilde{a} z_1^2 + z_1\overline{z}_1 + \overline{\gamma}\,\overline{z}_1^2 + O(|z_1|^3)$. By a qudratic unimodular transformation $(z_1, z_2) \mapsto (z_1, z_2 - (\tilde{a} - \gamma)z_1^2)$, we obtain

$$(3.1) \qquad M : \begin{cases} z_2 = q(z_1, \overline{z}_1) + H(z_1, \overline{z}_1), & H(z_1, \overline{z}_1) = O(|z_1|^3), \\ q(z_1, \overline{z}_1) = z_1\overline{z}_1 + \gamma z_1^2 + \overline{\gamma}\,\overline{z}_1^2, & \gamma \in \mathbb{C}. \end{cases}$$

We want to show that γ is a unimodular invariant. To see this, consider another surface

$$M' : z_2' = z_1'\overline{z'}_1 + \gamma' z_1'^2 + \overline{\gamma'}\,\overline{z'}_1^2 + O(|z_1'|^3).$$

Assume that there exists a unimodular transformation $F(z_1, z_2)$ such that $F(M) = M'$. Since $dF(0)$ preserves the z_1-axis, we have $F(z_1, z_2) = (az_1 + bz_2, cz_2) + O(2)$. From the defining functions of M and M', we get

$$c(z_1\overline{z}_1 + \gamma z_1^2 + \overline{\gamma}\,\overline{z}_1^2) = |a|^2 z_1\overline{z}_1 + a^2\gamma' z_1^2 + \overline{a}^2\overline{\gamma'}\,\overline{z}_1^2 + O(|z_1|^3).$$

Since $ac = 1$, the above equation implies that $c = 1 = a$ and $\gamma = \gamma'$. For later use, we note that under the assumption of $\gamma = \gamma'$, we have $\det(DF) \in \mathbb{R} \setminus \{0\}$ even without restricting F to be unimodular. For a biholomorphic change of coordinates, γ may be replaced by the Bishop invariant $|\gamma|$.

We now consider only cases $|\gamma| \neq 0, \frac{1}{2}$, or $1/2 < |\gamma| < \infty$ with non-exceptional values. We need the following:

Theorem 3.1 (Moser-Webster [5]). *Let M be a real analytic surface given by (3.1). Then M is formally equivalent to*

$$(3.2) \qquad Q_{\gamma,\epsilon,s} : x_2 = z_1\overline{z}_1 + (1 + \epsilon x_2^s)(\gamma z_1^2 + \overline{\gamma}\,\overline{z}_1^2), \qquad y_2 = 0,$$

of which s is a positive integer with $\epsilon = \pm 1$, or $\epsilon = 0\,(s = \infty)$. The full set of formal invariants of M near 0 is given by $\{|\gamma|, \epsilon, s\}$. Denote by $\mathrm{Aut}(Q_{\gamma,\epsilon,s})$ the group of formal automorphisms of $Q_{\gamma,\epsilon,s}$. Then $\mathrm{Aut}(Q_{\gamma,0,s})$ consists of transformations in the form:

$$(z_1, z_2) \mapsto (z_1 a(z_2), z_2 a^2(z_2)), \qquad a(z_2) = \overline{a}(z_2),\ a(0) \neq 0.$$

In the case $\epsilon \neq 0$, $\mathrm{Aut}(Q_{\gamma,\epsilon,s}) = \{1, \sigma \mid \sigma(z_1, z_2) = (-z_1, z_2)\}$.

4. Normal forms for unimodular transformations

We first discuss the formal theory of normal forms under unimodular transformations. Therefore, surfaces and transformations are given by formal power series.

Fix a formal surface $M \subset \mathbb{C}^2$. Assume that $0 \in M$ is a non-degenerate complex tangent. Then by a unimodular transformation we may assume that M is normalized into (3.1). As we have seen in section 3, γ in (3.1) is a unimodular invariant. Therefore, we need only seek a classification of surfaces which have the form (3.1) with a fixed γ. Denote by $\mathfrak{S}(M)$ the set of formal surfaces which are equivalent to M by formal transformations. Let $\mathfrak{U}$ be the group of formal unimodular transformations which preserve the quadratic part of (3.1). Then $\mathfrak{U}$ acts on $\mathfrak{S}(M)$ in the obvious way. Denote by $\mathfrak{S}(M)/\sim$ the equivalence classes of the action of $\mathfrak{U}$ on $\mathfrak{S}(M)$.

In section 3 we showed that if a biholormorphic mapping $z' = \phi(z)$ leaves $\mathfrak{S}(M)$ invariant, then $\det(D\phi)(0) \in \mathbb{R} \setminus \{0\}$. Let $\mathfrak{F}$ be all formal power series of complex coefficients with the constant term in $\mathbb{R} \setminus \{0\}$. Denote by $\mathrm{Aut}(M)$ the group of formal automorphisms of M, i.e., all the formal mappings $\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\psi(M) = (M)$. Consider the action of $\mathrm{Aut}(M)$ on $\mathfrak{F}$ defined by

$$(f, \psi) \mapsto f \circ \psi \cdot \det(D\psi), \qquad f \in \mathfrak{F}, \ \psi \in \mathrm{Aut}(M).$$

Let $\mathfrak{F}/\sim$ be the equivalence classes of action $\mathrm{Aut}(M)$ on $\mathfrak{F}$. For $M' \in \mathfrak{S}(M)$, we choose a formal mapping ϕ such that $\phi(M) = M'$. Set $\imath([M']) = [\det(D\phi)]$, in which [ ] stands for the equivalence class.

Lemma 4.1. *$\imath : \mathfrak{S}(M)/\sim \ \to \mathfrak{F}/\sim$ is bijective.*

Proof. Let ψ_1 and ψ_2 be two formal mappings such that $\psi_1(M)$ and $\psi_2(M) \in \mathfrak{S}(M)$. We first show that $\imath$ is well defined. Assume that there exists $\varphi \in \mathfrak{U} : \psi_1(M) \to \psi_2(M)$. Then $\phi = \psi_2^{-1}\varphi\psi_1 \in \mathrm{Aut}(M)$. Hence $\det(D(\psi_2\phi)) = \det(D\psi_1)$, which implies that $\det(D\psi_1) \equiv \det(D\psi_2)$ in $\mathfrak{F}/\sim$. To show $\imath$ is injective, assume that there is $\phi \in \mathrm{Aut}(M)$ such that $\det(D\psi_1) = \det(D(\psi_2\phi))$. Then $\varphi = \psi_2\phi^{-1}\psi_1^{-1} \in \mathfrak{U}$ maps $\psi_1(M)$ into $\psi_2(M)$. Clearly $\varphi \in \mathfrak{U}$, so $\psi_1(M) \equiv \psi_2(M)$ in $\mathfrak{S}(M)/\sim$. Conversely, for $f \in \mathfrak{F}$, we consider a mapping ψ given by

$$\psi(z_1, z_2) = (c\beta(z_1, z_2)z_1, c^2 z_2), \qquad c \in \mathbb{R} \setminus \{0\}, \ \beta(0) = 1. \tag{4.1}$$

Then ψ is uniquely determined by $\det(D\psi) = f$. Clearly, $\psi(M)$ is a realization for f. □

We need to identify $\mathfrak{F}/\sim$ with suitable normal forms.

Lemma 4.2. *Let $M = Q_{\gamma,\epsilon,s}$. Then for $f \in \mathfrak{F}$ there exists a unique $\phi \in \mathrm{Aut}(M)$ such that $\tilde{f} = f \circ \phi \cdot \det(D\phi) = r\alpha(z)$, while constant r and formal power series $\alpha(z)$ satisfy the following normalizing conditions:*

(a) *$r > 0$ and $\alpha(0) = 1$ for $\epsilon \neq 0$; or*

(b) *$r = 1$ and $\alpha(0, z_2) = e^{i\theta(z_2)}$, $\theta(0) = 0$, and $\overline{\theta}(z_2) = \theta(z_2)$, if $\epsilon = 0$.*

Proof. We need to apply Theorem 3.1. The proof for (a) is obvious, namely, we take $\phi = \mathrm{Id}$ for $f(0) > 0$, or $\phi(z_1, z_2) = (-z_1, z_2)$ if $f(0) < 0$.

For (b), write uniquely

$$f(0, z_2) = r_0(z_2)e^{i\theta_0(z_2)},$$

where both $r_0(z_2)$ and $\theta_0(z_2)$ have real coefficients and $\theta_0(0) = 0$. Let $\phi \in \mathrm{Aut}(M)$. Then from (c) in Theorem 3.1, we have

$$\phi(z_1, z_2) = (a(z_2)z_1, a^2(z_2)z_2)$$

for some power series $a(z_2)$ with real coefficients and $a(z_2) \neq 0$. Set

$$\begin{aligned}\tilde{f}(0, z_2) &= f \circ \phi \cdot \det(D\phi)(0, z_2) \\ &= f(0, a^2(z_2)z_2)(a^3(z_2) + 2z_2a^2(z_2)a'(z_2)).\end{aligned}$$

From the normalizing condition $\tilde{f}(0, z_2) = e^{i\theta(z_2)}$, we obtain a functional equation

$$r_0(a^2(z_2)z_2)(a^3(z_2) + 2z_2a^2(z_2)a'(z_2)) = 1, \tag{4.2}$$

where $a(z_2)$, $r_0(z_2)$ have real coefficients and non-zero constant terms.

We want to verify (b) by showing the existence and uniqueness of solution $a(z_2)$ to (4.2). To solve (4.2), rewrite

$$a(t) = a(0)(1 + A(t)) \quad \text{and} \quad r_0(a^2(0)t) = r_0(0)(1 + b(t)).$$

Then (4.2) becomes

$$\begin{cases} 3A(t) + 2tA'(t) = \dfrac{-b(t + tA(t))}{1 + b(t + tA(t))} \\ \qquad\qquad - (2A(t) + A^2(t))(A(t) + 2tA'(t)), \\ A(0) = 0 = b(0). \end{cases} \tag{4.3}$$

Let $A(t) = \sum_{k=1}^{\infty} A_k t^k$ and $b(t) = \sum_{k=1}^{\infty} b_k t^k$. Solving equation (4.3), we obtain

$$A_k = -\frac{b_k}{2k + 3} + P_k, \qquad k = 1, 2, \dots, \tag{4.4}$$

where $P_1 = 0$, and P_k is a polynomial in A_j, b_j $(j = 1, \dots, k - 1)$ with rational coefficients. (4.4) gives the unique solution to (4.2). □

Proof of Theorem 1.2. Assume that M is given by (3.1). From (b) of Theorem 3.1, there exists a formal mappping ψ such that $M = \psi(Q_{\gamma,\epsilon,s})$. Let $f = \det(D\psi)$. Applying Lemma 4.2, we have $f \equiv \tilde{f}$ in $\mathfrak{F}/\sim$, where $\tilde{f} = r\alpha(z)$, and r, α satisfy the normalizing condition stated in Lemma 4.2. Let

$$\psi(z_1, z_2) = (r^{\frac{1}{3}}\beta(z_1, z_2)z_1, r^{\frac{2}{3}}z_2),$$

in which β is determined by $\det(D(\psi)) = f$. Clearly, β still satisfies the normalizing condition stated in Lemma 4.2. In view of Lemma 4.1, M is equivalent to $\psi(Q_{\gamma,\epsilon,s})$ by formal unimodular transformations. Rewrite

$$\psi(z_1, z_2) = (r^{-\frac{1}{3}}\alpha(z_1, z_2)z_1, r^{-\frac{2}{3}}z_2).$$

It is easy to see that α satisfies the normalizing condition stated in Lemma 4.2. Clearly, $\psi(Q_{\gamma,\epsilon,s})$ is given by (1.4) with $\rho = \epsilon r^{-\frac{2}{3}s} \in \mathbb{R} \setminus \{0\}$ for $\epsilon \neq 0$ or $\rho = 0$ for $\epsilon = 0$.

To show the uniqueness of normalizing unimodular transformations, we assume that there exist unimodular transformations $\varphi_1, \varphi_2 : (M, 0) \to (\psi(Q_{\gamma,\epsilon,s}), 0)$. Let $\phi = \psi^{-1}\varphi_2\varphi_1^{-1}\psi$. Obviously, $\det(D\phi(0)) = 1$. The uniquness given by Lemma 4.2 implies that $\phi = \mathrm{Id}$ for the case $\epsilon \neq 0$. Hence $\varphi_1 = \varphi_2$. Next we assume that $\epsilon = 0$. We recall that $\phi \in \mathrm{Aut}(Q_{\gamma,\epsilon,s})$ is given by

$$\phi : (z_1, z_2) \mapsto (a(z_2)z_1, a^2(z_2)z_2). \tag{4.5}$$

Clearly, we have

$$\varphi_2\varphi_1^{-1}\psi\,|_{x_1=0}\colon z_2 \mapsto b(z_2)z_2,$$

in which $b(z_2)$ is a power series of real coefficients. We now have

$$\det(D\phi)(0, z_2) = \det(\psi^{-1}\varphi_2\varphi_1^{-1}\psi)(0, z_2) = e^{i(\theta(z_2)-\theta(b(z_2)z_2))}.$$

On the other hand, from $\phi(z_1, z_2) = (a(z_2)z_1, a^2(z_2)z_2)$, we get

$$e^{i(\theta(z_2)-\theta(b(z_2)z_2))} = a^3(z_2) + 2z_2a^2(z_2)a'(z_2).$$

Since θ, a, and b have real coefficients, we have $a \equiv 1 \equiv b$. Therefore $\varphi_1 = \varphi_2$. The theorem is proved. □

Next, we will deal with the problem of convergence. Let M be a real analytic surface as in Theorem 1.2. It is clear that if the normal form (1.4) is realized by convergent unimodular transformations, then M can also be transformed into the normal form of Moser-Webster. It is known that when the complex tangent is hyperbolic, the normal form (3.2) may not be realized by biholomorphic mappings [5]. Therefore, the unimodular transformation in Theorem 1.2 diverges in general when the hyperbolic complex tangent occurs. On the other hand, when the complex tangent is elliptic and $\gamma \neq 0$, M can indeed be transformed into (3.2) by biholomorphic mappings.

Proposition 4.3. *Let M be a real analytic surface as in Theorem* 1.2. *Then M can be transformed into* (3.2) *by biholomorphic transformations if and only if the unique unimodular transformation which transforms it into* (1.4) *is convergent.*

Proof. Assume M can be transformed into (3.2) by a biholomorphic mapping $\psi(z)$. It suffices to show that the function $f(z) = \det(D\psi)(z) \in \mathfrak{F}$ can be put into normal form stated in Lemma 4.2 through a convergent transformation in $\mathrm{Aut}(Q_{\gamma,\epsilon,s})$, which is trivial if M is not biholomorphically equivalent to the quadric $Q_\gamma \equiv Q_{\gamma,0,\infty}$. In the case M is biholomorphically equivalent to the quadric Q_γ, we need to show that (4.3) has a convergent solution $A(t)$, while $b(t)$ depends on $f(z_1, z_2)$ and converges near $t = 0$.

We need some notation. For two formal power series $p(t) = \sum_{k=0}^{\infty} p_k t^k$ and $q(t) = \sum_{k=0}^{\infty} q_k t^k$, let us denote

$$\widehat{p}(t) = \sum_{k=0}^{\infty} |p_k| t^k, \qquad [p]_k = p_k,$$

$$\sum_{k=0}^{\infty} p_k t^k \prec \sum_{k=0}^{\infty} q_k t^k, \quad \text{if } |p_k| \le q_k \quad \text{for all } j.$$

Comparing the coefficients in (4.3), we obtain

$$(2k+3)[A]_k = -\left[\frac{b(t+tA(t))}{1+b(t+tA(t))}\right]_k - \sum_{l=1}^{k-1}(2l+1)[A]_l \cdot [2A + A^2]_{k-l}.$$

Hence

$$\widehat{A(t)} \prec \frac{\widehat{b}(t+t\widehat{A}(t))}{1-\widehat{b}(t+t\widehat{A}(t))} + \widehat{A}(2\widehat{A}(t) + \widehat{A}^2(t)).$$

Consider the formal power series $A^*(t)$ defined by

$$(4.6) \qquad \begin{cases} A^*(t) = \dfrac{\widehat{b}(t + tA^*(t))}{1 - \widehat{b}(t + tA^*(t))} + A^*(2A^*(t) + A^{*2}(t)), \\ A^*(0) = 0. \end{cases}$$

By the implicit function theorem, $A^*(t)$ actually converges near $t = 0$. It is clear that $A(t) \prec A^*(t)$. Therefore, the solution to (4.3) converges, and the proof is complete. □

In [5], it is proved that a real analytic surface given by (3.1) with $0 < |\gamma| < 1/2$ can be transformed into the normal form (3.2) through biholomorphic mappings. Now Theorem 1.2 and Proposition 4.3 give

Corollary 4.4. *Let $M \subset \mathbb{C}^2$ be a real analytic surface with an elliptic complex tangent at 0. Assume that its Bishop invariant $|\gamma| \neq 0$. Then the unique formal unimodular transformation which transforms M into (1.4) is convergent.*

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References

1. V. I. Arnol'd, *On a characteristic class appearing in the quantization condition*, Functional Anal. Appl. **1** (1967), 1–14.
2. E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–22.
3. F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of* $\mathbb{C}^2$, Ann. Inst. Fourier (Grenoble) **37** (1987), 1–44.
4. J. K. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.
5. J. K. Moser and S. M. Webster, *Normal forms for real surfaces in* $\mathbb{C}^2$ *near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), 255–296.
6. J. Vey, *Sur le lemme de Morse*, Invent. Math. **40** (1977), 1–9.

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