THEOREMS OF CARATHEODORY
AND GLUSKIN FOR $0 < p < 1$

JESÚS BASTERO, JULIO BERNUÉS, AND ANA PEÑA

(Communicated by Dale Alspach)

Abstract. In this note we prove the $p$-convex analogue of both Caratheodory's convexity theorem and Gluskin's theorem concerning the diameter of Minkowski compactum.

Throughout this note $X$ will denote a real vector space and $p$ will be a real number, $0 < p < 1$. A set $A \subseteq X$ is called $p$-convex if $\lambda x + \mu y \in A$, whenever $x, y \in A$, and if $\lambda, \mu \geq 0$, with $\lambda^p + \mu^p = 1$. Given $A \subseteq X$, the $p$-convex hull of $A$ is defined as the intersection of all $p$-convex sets that contain $A$. This set is denoted by $p\text{-conv}(A)$. A (real) $p$-normed space $(X, \| \cdot \|)$ is a (real) vector space equipped with a quasi-norm such that $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, $\forall x, y \in X$. The unit ball of a $p$-normed space is a $p$-convex set and will be denoted by $B_X$.

We denote by $\mathcal{M}^p_n$ the class of all $n$-dimensional $p$-normed spaces. If $X, Y \in \mathcal{M}^p_n$, the Banach-Mazur distance $d(X, Y)$ is the infimum of the products $\|T\| \cdot \|T^{-1}\|$, where the infimum is taken over all the isomorphisms $T$ from $X$ onto $Y$. We shall use the notation and terminology commonly used in Banach space theory as it appears in [T-J].

The problem with which we are concerned is an aspect of the local structure of finite-dimensional $p$-Banach spaces. The well-known theorem of Gluskin gives a sharp lower bound of the diameter of the Minkowski compactum. In [G1] it is proved that $\text{diam}(\mathcal{M}^1_n) \geq cn$ for some absolute constant $c$. Our purpose is to study this problem in the $p$-convex setting. In [Pe], Peck gave an upper bound of the diameter of $\mathcal{M}^p_n$, namely, $\text{diam}(\mathcal{M}^p_n) \leq n^{2/p-1}$. We will show that this bound is optimal (Theorem 2). When proving it, in order to compute some volumetric estimates, it will be necessary to have the corresponding version for $p < 1$ of Caratheodory's convexity theorem (Theorem 1).

The results of this note are the following:

Received by the editors April 8, 1993.
1991 Mathematics Subject Classification. Primary 46A16, 46B20.
Key words and phrases. Minkowski compactum, $p$-convex sets.
The first and second authors were partially supported by Grant DGICYT PS 90-0120.
The third author was supported by Grant DGA (Spain).
Theorem 1. Let $A \subseteq \mathbb{R}^n$ and $0 < p < 1$. For every $x \in p\text{-conv}(A)$, $x \neq 0$, there exist linearly independent vectors $\{P_1, \ldots, P_k\} \subseteq A$ with $k \leq n$, such that $x \in p\text{-conv}\{P_1, \ldots, P_k\}$. Moreover, if $0 \in p\text{-conv}(A)$, there exists $\{P_1, \ldots, P_k\} \subseteq A$ with $k \leq n+1$ such that $0 \in p\text{-conv}\{P_1, \ldots, P_k\}$.

Theorem 2. Let $0 < p < 1$. There exists a constant $C_p > 0$ such that for every $n \in \mathbb{N}$

$$C_p n^{2/p-1} \leq \text{diam}(\mathcal{M}_n^p) \leq n^{2p-1}.$$  

Observe that Theorem 1 looks stronger than Caratheodory's one in the sense that we get $k \leq n$ and only $k \leq n+1$ can be assured for $p = 1$ (see [E, p. 35]). It will be clear that this is not so since vector 0 plays a particularly special role.

We begin by recalling the main property of $p$-convex hulls. It is probably known, but since we have not found it in any reference, we sketch its proof.

Lemma 1. Let $A \subseteq X$. The $p$-convex hull of $A$ coincides with the set of all finite sums $\sum \lambda_i x_i$ where $x_i$ are taken from $A$ (possibly with repetition), $\lambda_i \geq 0$, and $0 < \sum_i \lambda_i^p \leq 1$.

Proof. Straightforward arguments show that $p\text{-conv}(A)$ coincides with the set of all finite sums $\sum \lambda_i x_i$, $x_i \in A$, $\lambda_i \geq 0$, and $\sum \lambda_i^p = 1$. Now, we only have to prove that every nonzero element $x$ of the form $x = \sum_{i=1}^n \lambda_i x_i$, $x_i \in A$, $\sum_i \lambda_i^p < 1$ can be written as $x = \sum_{i=1}^m \mu_i y_i$, $y_i \in A$, $\sum \mu_i^p = 1$. Suppose $\lambda_i \neq 0$. Write $\lambda_i = \sum_{i=1}^k \beta_i$, with $\beta_i \geq 0$. We have $\sum_i \lambda_i^p < \sum_{i=1}^k \beta_i^p + \sum_{i=1}^n \lambda_i^p \leq k^{-1-p} \sum \lambda_i^p + \sum_i \lambda_i^p$. It is now clear, by a continuity argument, that we can find $k$ and $\beta_i \geq 0$, $1 \leq i \leq k$, such that $\lambda_i = \sum_{i=1}^k \beta_i$ and $\sum_i \lambda_i^p = 1$. The representation $x = \sum_{i=1}^k \beta_i x_i + \sum_{i=2}^n \lambda_i x_i$ does the job. □

Remark. Observe, in particular, that for every $0 \neq x \in X$, $p\text{-conv}\{x\} = (0, x] = \{\lambda x; 0 < \lambda \leq 1\}$. This situation is rather different from the case when $p = 1$.

Proof of Theorem 1. Let $x \in p\text{-conv}(A)$, $x \neq 0$. Let $N$ be the smallest integer so that $x$ in the $p$-convex hull of a subset $\{P_1, \ldots, P_N\}$ of $A$. Consider the set of all $(\alpha_i) \geq 0$ with $x = \sum_{i=1}^n \alpha_i P_i$, $0 < \sum_{i=1}^n \alpha_i^p \leq 1$. Minimize $\sum_{i=1}^N \alpha_i^p$ on this set, and denote the optimum by $(\lambda_i)$. Clearly $\lambda_i > 0$, for all $i = 1, \ldots, N$. Suppose $\{P_1, \ldots, P_N\}$ are linearly dependent; then there exists nontrivial coefficients $(\mu_i)$ so that $\sum_{i=1}^N \mu_i P_i = 0$. If $\delta > 0$ is small enough, all the coefficients $\lambda_i + t \mu_i > 0$ and the function $\phi(t) = \sum_{i=1}^N (\lambda_i + t \mu_i)^p$ defined for $t \in (-\delta, \delta)$ has a minimum in $t = 0$, which contradicts the fact that the second derivative of $\phi(t)$ is negative.

If $0 \in p\text{-conv}(A)$, then $0 = \sum_{i=1}^N \lambda_i P_i$, $P_i \in A$, $\lambda_i > 0$, $\forall i$, and $\sum_{i=1}^N \lambda_i^p = 1$. We can suppose $P_1, \ldots, P_m$ linearly independent with $m \leq n$. We consider $\sum_{i=1}^{m+1} \lambda_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$. If we apply the first part of the proof to $x = \sum_{i=1}^{m+1} \lambda_i s^{-1} P_i$, $s^p = \sum_{i=1}^{m+1} \lambda_i^p$, we obtain $\sum_{i=1}^m \beta_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$, with $\sum_{i=1}^m \beta_i^p \leq 1$. Hence $0 \leq p\text{-convex} \text{envelope of } N-1 \text{ points}$. Repeat the argument until reaching a representation of length $\leq n+1$. □
Next we are going to prove Theorem 2. The proof follows Gluskin’s original ideas. We first introduce some notation. $S^{n-1}$ will denote the euclidean sphere in $\mathbb{R}^n$ with its normalized Haar measure $\mu_{n-1}$ and $\Omega$ will be the product space $S^{n-1} \times \cdots \times S^{n-1}$ endowed with the product probability $\mathbb{P}$. If $K \subset \mathbb{R}^n$, $|K|$ is the Lebesgue measure of $K$. If $A = (P_1, \ldots, P_n) \subset \Omega$, we write $Q_p(A) = p$-conv\{$\pm e_i, \pm P_i | 1 \leq i \leq n$\}, $\{e_i\}_{i=1}^n$ being the canonical basis of $\mathbb{R}^n$. We denote by $\|\cdot\|_{Q_p(A)}$ the $p$-norm in $\mathbb{R}^n$ whose unit ball is $Q_p(A)$.

We only need to prove that for some absolute constant $C_p > 0$, there exist $A, A' \in \Omega$ such that simultaneously both $\|T\|_{Q_p(A) \rightarrow Q_p(A')} \geq C_p n^{1/p - 1/2}$ and $\|T^{-1}\|_{Q_p(A') \rightarrow Q_p(A)} \geq C_p n^{1/p - 1/2}$ hold for any $T \in SL(n)$ (that is, any linear isomorphism in $\mathbb{R}^n$ with det $T = 1$).

Straightforward arguments show that it is enough to see that for any $A' \in \Omega$, $\mathbb{P}\{A \in \Omega | \|T\|_{Q_p(A) \rightarrow Q_p(A')} < C_p n^{1/p - 1/2} \text{ for some } T \in SL(n)\} < \frac{1}{2}$.

Fix $A' \in \Omega$ and $t > 0$, and write $\Omega(A', t) = \{A \in \Omega | \|T\|_{Q_p(A) \rightarrow Q_p(A')} < t \text{ for some } T \in SL(n)\}$.

The proof of the following lemma is analogous to the one in the case $p = 1$ (see [T-J], §38).

**Lemma 2.** Let $A' \in \Omega$ and $t > 0$.

(i) There exists a $t^p$-net $N(A', t)$ in $\{T \in SL(n) | \|T\|_{l_2^p \rightarrow Q_p(A')} \leq t\}$ with respect to the metric induced by $\|\cdot\|_{l_2^p \rightarrow Q_p(A')}^p$. The cardinality $|N(A', t)| \leq \left(3t^{-1/p} n^{1/p - 1/2}\right)^n$.

(ii) $\Omega(A', t) \subset \bigcup_{T \in N(A', t)} \{A \in \Omega | \|T(P_i)\|_{Q_p(A')} \leq 2^{1/p} t, \forall P_i \in A\}$.

(iii) Given $T \in SL(n)$,

$\mathbb{P}\{A \in \Omega | \|T(P_i)\|_{Q_p(A')} \leq 2^{1/p} t, \forall P_i \in A\} \leq (2^{1/p} t)^n \left(\frac{|Q_p(A')|}{|B_{l_2^n}|}\right)^n$.

**Proof of Theorem 2.** Numerical constants are always denoted by the same letters $C$ (or $C_p$, if it depends only on $p$), although they may have different values from line to line. Using consecutively the three preceding lemmas we have for every $A' \in \Omega$ and $t > 0$,

$\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/p - 1/2})^n \left(\frac{|Q_p(A')|^2}{|B_{l_2^n}|^n \cdot \{T \in SL(n) | \|T\|_{l_2^p \rightarrow l_2^p} \leq 1}\right)$.

It is well known that for some absolute constant $C > 0$ (see [T-J]), we have $|\{T \in SL(n) | \|T\|_{l_2^p \rightarrow l_2^p} \leq 1\}| \geq C^n |B_{l_2^n}|^n$.

Let $A' = \{P_1, \ldots, P_n\}$. By Theorem 1, $Q_p(A') \subset \cup p$-conv\{$P_{k_1}, \ldots, P_{k_n}$\} where the union runs over the $\binom{4n}{n}$ choices of $\{P_{k_i}\}_{i=1}^n \subset \{\pm e_i, \pm P_i | 1 \leq i \leq n\}$. Since $\|P_i\|_2 = 1$ and $|p$-conv\{$P_{k_1}, \ldots, P_{k_n}$\}| is equal to $|\det(P_{k_1}, \ldots, P_{k_n})| \cdot |p$-conv\{$e_1, \ldots, e_n$\}|, we get $|Q_p(A')| \leq \left(\binom{4n}{n}\right)^n |B_{l_2^n}|^{2n} \leq C_p n^{-n} p^{n/2} 2^{-n}$ for some constant $C_p$ (see [Pi, p. 11]). Hence $\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/2 - 1/p})^n$. If we take a suitable $t > 0$, we can assure $\mathbb{P}(\Omega(A', t)) < \frac{1}{2}$ and the result follows. □
Remark. With straightforward variations in the proof we can state the following result: Given $0 < p, q < 1$, for any natural number $N$ we can find two $aN$-dimensional quotients of $l_p^N$ having Banach-Mazur distance greater than or equal to $C(p, a)N^{2/p-1}$.

Remark. Given a $p$-normed space $X$ and $p < q < 1$, we define the $q$-Banach envelope of $X$ as the $q$-normed space, $X^q$, whose unit ball is the $q$-convex envelope of the unit ball of $X$. It is easy to see that $d(X, X^q) \leq d(X, Y)$ for any $n$-dimensional $q$-normed space $Y$ (see [Pe, GK]). Theorem 1 shows that $d(X, X^q) \leq n^{1/p-1/q}$. Indeed, for every $x \in B_{X^q}$, $\|x\|_{X^q} = 1$, there exist $P_1, \ldots, P_n \in B_X$ such that $x = \sum_{i=1}^n \lambda_i P_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i^q \leq 1$, and $1 \leq \|x\|_X \leq \sum_{i=1}^n \lambda_i^p \|P_i\|_X \leq \sum_{i=1}^n \lambda_i^p \leq n^{1/p-1/q}$; by homogeneity we achieve the result. Now it is easy to see that if $X$, $Y$ are the spaces appearing in Theorem 2, then $d(X, X^q) \geq C_p n^{1/p-1/q}$, $d(Y, Y^q) \geq C_p n^{1/p-1/q}$, and $d(X^q, Y^q) \geq C_p n^{2/q-1}$. In particular, for $q = 1$, $d(X, X^1) \geq C_p n^{1/p-1}$, $d(Y, Y^1) \geq C_p n^{1/p-1}$, and $d(X^1, Y^1) \geq C_p n$.

ACKNOWLEDGMENTS

The authors are indebted to the referee for showing them the simpler proof of Theorem 1 which notably simplifies a previous one.

REFERENCES


