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THE THEOREMS OF CARATHEODORY AND GLUSKIN FOR $0 < p < 1$

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ABSTRACT. In this note we prove the p-convex analogue of both Caratheodory's convexity theorem and Gluskin's theorem concerning the diameter of Minkowski compactum.

Throughout this note X will denote a real vector space and p will be a real number, $0 < p < 1$. A set $A \subset X$ is called p-convex if $\lambda x + \mu y \in A$, whenever $x, y \in A$, and if $\lambda, \mu \geq 0$, with $\lambda^p + \mu^p = 1$. Given $A \subseteq X$, the p-convex hull of A is defined as the intersection of all p-convex sets that contain A. This set is denoted by $p\text{-}\operatorname{conv}(A)$. A (real) p-normed space $(X, \|\cdot\|)$ is a (real) vector space equipped with a quasi-norm such that $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, $\forall x, y \in X$. The unit ball of a p-normed space is a p-convex set and will be denoted by B_X.

We denote by $\mathscr{M}_n^p$ the class of all n-dimensional p-normed spaces. If $X, Y \in \mathscr{M}_n^p$, the Banach-Mazur distance $d(X, Y)$ is the infimun of the products $\|T\| \cdot \|T^{-1}\|$, where the infimun is taken over all the isomorphisms T from X onto Y. We shall use the notation and terminology commonly used in Banach space theory as it appears in [T-J].

The problem with which we are concerned is an aspect of the local structure of finite-dimensional p-Banach spaces. The well-known theorem of Gluskin gives a sharp lower bound of the diameter of the Minkowski compactum. In [G1] it is proved that $\operatorname{diam}(\mathscr{M}_n^1) \geq cn$ for some absolute constant c. Our purpose is to study this problem in the p-convex setting. In [Pe], Peck gave an upper bound of the diameter of $\mathscr{M}_n^p$, namely, $\operatorname{diam}(\mathscr{M}_n^p) \leq n^{2/p-1}$. We will show that this bound is optimal (Theorem 2). When proving it, in order to compute some volumetric estimates, it will be necessary to have the corresponding version for $p < 1$ of Caratheodory's convexity theorem (Theorem 1).

The results of this note are the following:

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Theorem 1. *Let $A \subseteq \mathbb{R}^n$ and $0 < p < 1$. For every $x \in p\text{-conv}(A)$, $x \neq 0$, there exist linearly independent vectors $\{P_1, \dots, P_k\} \subseteq A$ with $k \leq n$, such that $x \in p\text{-conv}\{P_1, \dots, P_k\}$. Moreover, if $0 \in p\text{-conv}(A)$, there exists $\{P_1, \dots, P_k\} \subseteq A$ with $k \leq n+1$ such that $0 \in p\text{-conv}\{P_1, \dots, P_k\}$.*

Theorem 2. *Let $0 < p < 1$. There exists a constant $C_p > 0$ such that for every $n \in \mathbb{N}$*

$$C_p n^{2/p-1} \leq \operatorname{diam}(\mathscr{M}_n^p) \leq n^{2p-1}.$$

Observe that Theorem 1 looks stronger than Caratheodory's one in the sense that we get $k \leq n$ and only $k \leq n+1$ can be assured for $p = 1$ (see [E, p. 35]). It will be clear that this is not so since vector 0 plays a particularly special role.

We begin by recalling the main property of p-convex hulls. It is probably known, but since we have not found it in any reference, we sketch its proof.

Lemma 1. *Let $A \subset X$. The p-convex hull of A coincides with the set of all finite sums $\sum \lambda_i x_i$ where x_i are taken from A (possibly with repetition), $\lambda_i \geq 0$, and $0 < \sum \lambda_i^p \leq 1$.*

Proof. Straightforward arguments show that $p\text{-conv}(A)$ coincides with the set of all finite sums $\sum \lambda_i x_i$, $x_i \in A$, $\lambda_i \geq 0$, and $\sum \lambda_i^p = 1$. Now, we only have to prove that every nonzero element x of the form $x = \sum_{i=1}^n \lambda_i x_i$, $x_i \in A$, $\sum_{i=1}^n \lambda_i^p < 1$ can be written as $x = \sum_{i=1}^m \mu_i y_i$, $y_i \in A$, $\sum_1^m \mu_i^p = 1$. Suppose $\lambda_1 \neq 0$. Write $\lambda_1 = \sum_{i=1}^k \beta_i$, with $\beta_i \geq 0$. We have $\sum_{i=1}^n \lambda_i^p \leq \sum_{i=1}^k \beta_i^p + \sum_{i=2}^n \lambda_i^p \leq k^{1-p}\lambda_1^p + \sum_{i=2}^n \lambda_i^p$. It is now clear, by a continuity argument, that we can find k and $\beta_i \geq 0$, $1 \leq i \leq k$, such that $\lambda_1 = \sum_{i=1}^k \beta_i$ and $\sum_{i=1}^k \beta_i^p + \sum_{i=2}^n \lambda_i^p = 1$. The representation $x = \sum_{i=1}^k \beta_i x_i + \sum_{i=2}^n \lambda_i x_i$ does the job. □

Remark. Observe, in particular, that for every $0 \neq x \in X$, $p\text{-conv}\{x\} = (0, x] = \{\lambda x ; 0 < \lambda \leq 1\}$. This situation is rather different from the case when $p = 1$.

Proof of Theorem 1. Let $x \in p\text{-conv}(A)$, $x \neq 0$. Let N be the smallest integer so that x in the p-convex hull of a subset $\{P_1, \dots, P_N\}$ of A. Consider the set of all $(\alpha_i) \geq 0$ with $x = \sum_{i=1}^N \alpha_i P_i$, $0 < \sum_{i=1}^N \alpha_i^p \leq 1$. Minimize $\sum_{i=1}^N \alpha_i^p$ on this set, and denote the optimum by (λ_i). Clearly $\lambda_i > 0$, for all $i = 1, \dots, N$. Suppose $\{P_1, \dots, P_N\}$ are linearly dependent; then there exists nontrivial coefficients (μ_i) so that $\sum_{i=1}^N \mu_i P_i = 0$. If $\delta > 0$ is small enough, all the coefficients $\lambda_i + t\mu_i > 0$ and the function $\phi(t) = \sum_{i=1}^N (\lambda_i + t\mu_i)^p$ defined for $t \in (-\delta, \delta)$ has a minimum in $t = 0$, which contradicts the fact that the second derivative of $\phi(t)$ is negative.

If $0 \in p\text{-conv}(A)$, then $0 = \sum_{i=1}^N \lambda_i P_i$, $P_i \in A$, $\lambda_i > 0$, $\forall i$, and $\sum_{i=1}^N \lambda_i^p = 1$. We can suppose $P_1, \dots, P_m$ linearly independent with $m \leq n$. We consider $\sum_{i=1}^{m+1} \lambda_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$. If we apply the first part of the proof to $\tilde{x} = \sum_{i=1}^{m+1} \lambda_i s^{-1} P_i$, $s^p = \sum_{i=1}^{m+1} \lambda_i^p$, we obtain $\sum_{i=1}^m \beta_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$, with $\sum_{i=1}^m \beta_i^p \leq 1$. Hence $0 \leq p$-convex envelope of $N-1$ points. Repeat the argument until reaching a representation of length $\leq n+1$. □

Next we are going to prove Theorem 2. The proof follows Gluskin's original ideas. We first introduce some notation. S^{n-1} will denote the euclidean sphere in $\mathbb{R}^n$ with its normalized Haar measure μ_{n-1} and Ω will be the product space $S^{n-1} \times \overset{(n)}{\cdots} \times S^{n-1}$ endowed with the product probability $\mathbb{P}$. If $K \subset \mathbb{R}^n$, $|K|$ is the Lebesgue measure of K. If $A = (P_1, \dots, P_n) \subset \Omega$, we write $Q_p(A) = p\text{-conv}\{\pm e_i, \pm P_i | 1 \leq i \leq n\}$, $\{e_i\}_{i=1}^n$ being the canonical basis of $\mathbb{R}^n$. We denote by $\|\cdot\|_{Q_p(A)}$ the p-norm in $\mathbb{R}^n$ whose unit ball is $Q_p(A)$.

We only need to prove that for some absolute constant $C_p > 0$, there exist $A, A' \in \Omega$ such that simultaneously both $\|T\|_{Q_p(A) \to Q_p(A')} \geq C_p n^{1/p-1/2}$ and $\|T^{-1}\|_{Q_p(A') \to Q_p(A)} \geq C_p n^{1/p-1/2}$ hold for any $T \in \mathrm{SL}(n)$ (that is, any linear isomorphism in $\mathbb{R}^n$ with $\det T = 1$).

Straightforward arguments show that it is enough to see that for any $A' \in \Omega$, $\mathbb{P}\{A \in \Omega | \|T\|_{Q_p(A) \to Q_p(A')} < C_p n^{1/p-1/2}$ for some $T \in \mathrm{SL}(n)\} < \frac{1}{2}$.

Fix $A' \in \Omega$ and $t > 0$, and write $\Omega(A', t) = \{A \in \Omega | \|T\|_{Q_p(A) \to Q_p(A')} < t$ for some $T \in \mathrm{SL}(n)\}$.

The proof of the following lemma is analogous to the one in the case $p = 1$ (see [T-J], §38]).

Lemma 2. *Let $A' \in \Omega$ and $t > 0$.*

(i) *There exists a t^p-net $N(A', t)$ in $\{T \in \mathrm{SL}(n) | \|T\|_{l_p^n \to Q_p(A')} \leq t\}$ with respect to the metric induced by $\|\cdot\|^p_{l_2^n \to Q_p(A')}$ of cardinality*

$$|N(A', t)| \leq (3^{1/p} n^{1/p-1/2})^{n^2} \frac{|Q_p(A')|^n}{|\{T \in \mathrm{SL}(n) | \|T\|_{l_2^n \to l_2^n} \leq 1\}|}.$$

(ii)

$$\Omega(A', t) \subseteq \bigcup_{T \in N(A', t)} \{A \in \Omega | \|T(P_i)\|_{Q_p(A')} \leq 2^{1/p} t, \ \forall P_i \in A\}.$$

(iii) *Given $T \in \mathrm{SL}(n)$,*

$$\mathbb{P}\{A \in \Omega | \|T(P_i)\|_{Q_p(A')} \leq 2^{1/p} t, \ \forall P_i \in A\} \leq (2^{1/p} t)^{n^2} \left(\frac{|Q_p(A')|}{|B_{l_2^n}|} \right)^n.$$

Proof of Theorem 2. Numerical constants are always denoted by the same letters C (or C_p, if it depends only on p), although they may have different values from line to line. Using consecutively the three preceding lemmas we have for every $A' \in \Omega$ and $t > 0$,

$$\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/p-1/2})^{n^2} \frac{|Q_p(A')|^{2n}}{|B_{l_2^n}|^n \cdot |\{T \in \mathrm{SL}(n) | \|T\|_{l_2^n \to l_2^n} \leq 1\}|}.$$

It is well known that for some absolute constant $C > 0$ (see [T-J]), we have $|\{T \in \mathrm{SL}(n) | \|T\|_{l_2^n \to l_2^n} \leq 1\}| \geq C^{n^2} |B_{l_2^n}|^n$.

Let $A' = \{P_1, \dots, P_n\}$. By Theorem 1, $Q_p(A') \subseteq \bigcup p\text{-conv}\{P_{k_1}, \dots, P_{k_n}\}$ where the union runs over the $\binom{4n}{n}$ choices of $\{P_{k_i}\}_{i=1}^n \subseteq \{\pm e_i, \pm P_i, 1 \leq i \leq n\}$. Since $\|P_i\|_2 = 1$ and $|p\text{-conv}\{P_{k_1}, \dots, P_{k_n}\}|$ is equal to $|\det[P_{k_1}, \dots, P_{k_n}]| \cdot |p\text{-conv}\{e_1, \dots, e_n\}|$, we get $|Q_p(A')| \leq \binom{4n}{n} |B_{l_p^n}|/2^n \leq C_p^n n^{-n/p} 2^{-n}$ for some constant C_p (see [Pi, p. 11]). Hence $\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/2-1/p})^{n^2}$. If we take a suitable $t > 0$, we can assure $\mathbb{P}(\Omega(A', t)) < \frac{1}{2}$ and the result follows. □

Remark. With straightforward variations in the proof we can state the following result: Given $0 < p \leq 1$ and $0 < \alpha < 1$, there exists a constant $0 < C(p, \alpha) < 1$ such that for any natural number N we can find two αN-dimensional quotients of l_p^N having Banach-Mazur distance greater than or equal to $C(p, \alpha)N^{2/p-1}$.

Remark. Given a p-normed space X and $p < q \leq 1$, we define the q-Banach envelope of X as the q-normed space, X^q, whose unit ball is the q-convex envelope of the unit ball of X. It is easy to see that $d(X, X^q) \leq d(X, Y)$ for any n-dimensional q-normed space Y (see [Pe, GK]). Theorem 1 shows that $d(X, X^q) \leq n^{1/p-1/q}$. Indeed, for every $x \in B_{X^q}$, $\|x\|_{X^q} = 1$, there exist $P_1, \dots, P_n \in B_X$ such that $x = \sum_{i=1}^n \lambda_i P_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i^q \leq 1$, and $1 \leq \|x\|_X \leq \sum_{i=1}^n \lambda_i^p \|P_i\|_X^p \leq \sum_{i=1}^n \lambda_i^p \leq n^{1/p-1/q}$; by homogeneity we achieve the result. Now it is easy to see that if X, Y are the spaces appearing in Theorem 2, then $d(X, X^q) \geq C_p n^{1/p-1/q}$, $d(Y, Y^q) \geq C_p n^{1/p-1/q}$ and $d(X^q, Y^q) \geq C_p n^{2/q-1}$. In particular, for $q = 1$, $d(X, X^1) \geq C_p n^{1/p-1}$, $d(Y, Y^1) \geq C_p n^{1/p-1}$, and $d(X^1, Y^1) \geq C_p n$.

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