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UNIFORM RATIONAL APPROXIMATION

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ABSTRACT. Let K be a compact subset of the complex plane $\mathbb{C}$, and let $P(K)$ and $R(K)$ be the closures in $C(K)$ of polynomials and rational functions with poles off K, respectively. Suppose that $R(K) \neq C(K)$, λ is a nonpeak point for $R(K)$, and g is continuous on $\mathbb{C}$ and C^1 in a neighborhood of λ. Then $P(K)g + R(K)$ is not dense in $C(K)$. In fact, our proof shows that there are a lot of smooth functions which are not in the closure of $P(K)g + R(K)$.

For a compact subset K of the complex plane let $C(K)$ denote the Banach algebra of complex-valued continuous functions on K with customary norm. Let $P(K)$ and $R(K)$ denote the closures in $C(K)$ of analytic polynomials and rational functions with poles off K, respectively. In [T2], Thomson proved the following interesting result.

Thomson's Theorem. *If $R(K) \neq C(K)$, then $\overline{z}P(K) + R(K)$ is not dense in $C(K)$.*

The purpose of this paper is to show that Thomson's method works in more general cases. We prove the following generalized result.

Main Theorem. *Assume that $R(K) \neq C(K)$ and λ is a nonpeak point for $R(K)$. Let g be a continuous function on $\mathbb{C}$. Suppose that $\overline{\partial} g$ ($\overline{\partial}$ is the Cauchy-Riemann operator in the plane) is continuous in a neighborhood of λ. Then $P(K)g + R(K)$ is not dense in $C(K)$.*

Before going further, we need to introduce some notation. For a compactly supported complex Borel measure μ the Cauchy transform of μ is the function $\hat{\mu}$ defined by

$$\hat{\mu}(w) = \int (z-w)^{-1}\, d\mu(z).$$

This function is locally integrable with respect to the area measure dA. If a function f is analytic at ∞, then f can be represented by its Laurent series

$$f(z) = f(\infty) + a_1(z-z_0)^{-1} + a_2(z-z_0)^{-2} + \cdots$$

in a neighborhood of infinite. We define $f'(\infty)$ to be a_1 and $\beta(f, z_0)$ to be a_2. The number $f'(\infty)$ does not depend on z_0, but $\beta(f, z_0)$ does depend on the choice of z_0.

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We fix a compactly supported function g and a compact subset K. A closed square S with edge length d is called light if

$$\operatorname{dist}(g, R(S \cap K)) \leq d^4.$$

A square is heavy if it is not light.

We will borrow the coloring scheme and notation from §2 of [T2]. We shall let C, c be positive numbers, that may change from one appearance to the next.

Let ϕ be a smooth function with compact support; the localization operator T_ϕ is defined by

$$(T_\phi f)(\lambda) = \frac{1}{\pi} \int\int \frac{f(z) - f(\lambda)}{z - \lambda} \frac{\partial \phi}{\partial \overline{z}} \, dA(z)$$

where f is a continuous function on $\mathbb{C}_\infty$. One can easily prove the following norm estimation for T_ϕ:

$$\|T_\phi f\| \leq C\|f\| \operatorname{diameter}(\operatorname{supp} \phi)\|\overline{\partial}\phi\|.$$

Therefore, the operator T_ϕ is a bounded linear operator on $C(K)$. Let M be the space of finite complex Borel measures $(= C(K)^*)$. Then T_ϕ^* is a bounded linear operator on M. Moreover, if $\mu \in M$, then

$$\|T_\phi^* \mu\| \leq C \operatorname{diameter}(\operatorname{supp} \phi)\|\overline{\partial}\phi\|\|\mu\|$$

and

$$\int T_\phi f \, d\mu = \int f \, dT_\phi^* \mu.$$

Consequently, $T_\phi^* \mu \perp R(K)$ for each measure $\mu \perp R(K)$. For basic facts of T_ϕ, see [G].

Lemma 1. *Let g be a C^1 function with compact support. Then $g \in R(K)$ if and only if $\overline{\partial} g = 0$ on the set of nonpeak points for $R(K)$.*

Proof. Using Theorem 3.2.9 [B, p. 166] and Theorem 3.3.4 [B, p. 173], we easily prove this lemma.

Lemma 2. *Let g be a C^1 function with compact support and $|\overline{\partial} g(z)| \geq c > 0$ in an open disk O containing K, and let S be a light square with center 0 ($\in \widehat{K}$: polynomial convex hull of K) and edge length d (for d small enough). Let α and β be two complex numbers such that $|\alpha| \leq 1$ and $|\beta| \leq 1$. Then there exists a function f in $C(\mathbb{C}_\infty)$ such that the following hold:*

(1) $\|f\| \leq C$;
(2) f *is analytic off* S;
(3) $f(\infty) = 0$;
(4) $f'(\infty) = \alpha d$;
(5) $\beta(f, 0) = \beta d^2$;
(6) $|\int f \, d\mu| \leq C d^3 \|\mu\|$ *for each measure* $\mu \perp R(K)$.

Proof. Let ϕ be a smooth function supported in the disk of radius $\frac{d}{4}$ centered at zero such that:

(a) $0 \leq \phi \leq 1$;
(b) $\|\overline{\partial}\phi\| \leq C\frac{1}{d}$;
(c) $\int \phi \, dA = \frac{d^2}{8}$.

Let $\phi_1 = \phi(z + \frac{d}{4})$ and $\phi_2 = \phi(z - \frac{d}{4})$. Let $f_j = T_{\phi_j} g$ for $j = 1, 2$. Then

$$f'_j(\infty) = -\frac{1}{\pi}\int g\overline{\partial}\phi_j\,dA = \frac{1}{\pi}\int \overline{\partial}g\phi_j\,dA$$

and

$$\beta(f_j, 0) = -\frac{1}{\pi}\int zg\overline{\partial}\phi_j\,dA = \frac{1}{\pi}\int z\overline{\partial}g\phi_j\,dA.$$

There is a constant $\delta > 0$ such that for z_1, $z_2 \in \overline{O}$ and $|z_1 - z_2| < \delta$ (we may assume that $d < 2\delta$)

$$|\overline{\partial}g(z_1) - \overline{\partial}g(z_2)| < \varepsilon = \frac{c^2}{16\|\overline{\partial}g\|}.$$

Set

$$D = \int \overline{\partial}g\phi_1\,dA \int z\overline{\partial}g\phi_2\,dA - \int \overline{\partial}g\phi_2\,dA \int z\overline{\partial}g\phi_1\,dA.$$

We have the following computation:

$$\begin{aligned}
&\int \phi_1\,dA \int z\phi_2\,dA - \int \phi_2\,dA \int z\phi_1\,dA \\
&\quad = \int \phi\,dA \int \left(z + \frac{d}{4}\right)\phi\,dA - \int \phi\,dA \int \left(z - \frac{d}{4}\right)\phi\,dA \\
&\quad = \frac{d}{2}\left(\int \phi\,dA\right)^2 = \frac{d^5}{128}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|\int \overline{\partial}g\phi_1\,dA \int z\overline{\partial}g\phi_2\,dA - \int \overline{\partial}g(0)\phi_1\,dA \int z\overline{\partial}g\phi_2\,dA\right| \\
&\qquad \le \varepsilon \int \phi_1\,dA \int |z\overline{\partial}g|\phi_2\,dA \le \frac{d^5}{1024}c^2.
\end{aligned}$$

It follows from last two inequalities that

$$\left|D - \overline{\partial}g(0)^2\frac{d^5}{128}\right| \le 4\frac{d^5}{1024}c^2 = \frac{d^5}{256}c^2.$$

Hence, $|D| \ge \frac{d^5}{256}c^2$. Let

$$x = \pi\frac{d\int z\overline{\partial}g\phi_2\,dA}{D}, \qquad y = -\pi\frac{d\int z\overline{\partial}g\phi_1\,dA}{D},$$

$$u = -\pi\frac{d^2\int \overline{\partial}g\phi_2\,dA}{D}, \qquad w = \pi\frac{d^2\int \overline{\partial}g\phi_1\,dA}{D}.$$

Thus,

$$|x|, |y|, |u|, |w| \le C\frac{1}{d}.$$

Using Green's formula, we see that

$$T_{\phi_j}g(\lambda) = -\frac{1}{\pi}\int \frac{\overline{\partial}g\phi_j}{z - \lambda}\,dA.$$

Hence, $\|T_{\phi_j} g\| \le Cd$. Suppose that g_1 and g_2 are as follows:

$$g_1 = x f_1 + y f_2 \quad \text{and} \quad g_2 = u f_1 + w f_2 .$$

Then $\|g_1\|$, $\|g_2\| \le C$. Set $f = \alpha g_1 + \beta g_2$. It is easy to check that f satisfies (1)–(5). For (6), we have the following estimation:

$$\begin{aligned}\left|\int g_j \, d\mu\right| &\le C\frac{1}{d}\left|\int f_1 \, d\mu\right| + C\frac{1}{d}\left|\int f_2 \, d\mu\right| \\ &\le C\frac{1}{d}\left|\int g \, dT^*_{\phi_1}\mu\right| + C\frac{1}{d}\left|\int g dT^*_{\phi_2}\mu\right| \\ &\le \left(C\frac{1}{d}\|T^*_{\phi_1}\mu\| + C\frac{1}{d}\|T^*_{\phi_2}\mu\|\right) \operatorname{dist}(g, R(S \cap K)) \\ &\le Cd^3\|\mu\| .\end{aligned}$$

Using the last lemma and the same techniques in §3 of [T], one obtains the following lemma.

Lemma 3. *Let* δ, $\varepsilon > 0$ *and* $|\alpha|, |\beta| \le 1$. *Let* D *be the open disk with center* 0 *and radius* δ. *Suppose that there exists a sequence of light squares from* 0 *to* ∞ (*see* [T2] *for definitions*). *Then there exists* f *in* $C(\mathbb{C}_\infty)$ *such that the following hold*:

(1) $\|f\| \le C$;
(2) f *is analytic off* D;
(3) $f(\infty) = 0$;
(4) $f'(\infty) = \alpha\delta$;
(5) $\beta(f, 0) = \beta\delta^2$;
(6) $|\int f \, d\mu| \le \varepsilon\|\mu\|$ *for each measure* $\mu \perp R(K)$.

Theorem 4. *Suppose that* $R(K) \ne C(K)$ *and* λ *is a nonpeak point for* $R(K)$. *Let* $g \in C(\mathbb{C}_\infty)$ *have continuous first derivatives in a neighborhood of* λ *and* $\bar\partial g(\lambda) \ne 0$. *Let* U *be the set of points around which there is a sequence of heavy barriers* (*see* [T2] *for definitions*). *Then* $\lambda \in U$.

Proof. Choose $\delta_0 > 0$ so that $|\bar\partial g(z)| \ge c > 0$ on $O(\lambda, \delta_0)$. Suppose λ is not in U. Then there exists a sequence of light squares from λ to ∞ for $R(K \cap \overline{O(\lambda, \delta_0/2)})$. Therefore, we may restrict our attention to the algebra $R(K \cap \overline{O(\lambda, \delta_0/2)})$.

We use a similar argument in the proof of Theorem 4.11 of [T]. Let ε be δ^2 and $\alpha = 1$ and $\beta = 0$. Using Lemma 3, we have functions f_δ satisfying conditions (1)–(6). Let $g_\delta(z) = (z-\lambda)f_\delta(z)/f'_\delta(\infty)$. Using Maximum Modulus Principle, one easily sees that $\|g_\delta\| \le C$. A normal family argument shows that there exists a sequence of $\{\delta_n\}$ such that

$$g_{\delta_n}(z) \to f(z) = \begin{cases} 1, & z \ne \lambda, \\ 0, & z = \lambda, \end{cases}$$

pointwisely when $\delta_n \to 0$.

From Theorem 11.6 of [G, p. 56], one can construct a measure ν carried on the set of peak points for $R(K \cap \overline{O(\lambda, \delta_0/2)})$ such that for each rational function r with poles off $K \cap \overline{O(\lambda, \delta_0/2)}$ we have $r(\lambda) = \int r \, d\nu$. Let $\mu = \nu - \delta_\lambda$.

Then $\mu \perp R(K \cap \overline{O(\lambda, \delta_0/2)})$ and $\mu\{\lambda\} = -1$ since λ is a nonpeak point for $R(K \cap \overline{O(\lambda, \delta_0/2)})$ (see [G, Theorem 4.5, p. 205]). Using Lemma 3 again, we get

$$\left|\int g_{\delta_n}\, d\mu\right| = \frac{1}{\delta_n}\left|\int f_{\delta_n}((z-\lambda)\, d\mu)\right| \leq \frac{\varepsilon_n}{\delta_n}\|(z-\lambda)\mu\| \leq C\delta_n\|\mu\|.$$

It follows from Lebesgue's dominating theorem that $\int f\, d\mu = 0$. Hence,

$$\int (1-f)\, d\mu = \mu\{\lambda\} = 0.$$

This is a contradiction.

Using the same proof as in §4 of [T2], one can show the following theorem.

Theorem 5. *Let g be as in the main theorem. From the last theorem, we see that $U \neq \varnothing$. Then for each $\lambda \in U$ there exists a measure μ_λ such that $\mu_\lambda \perp R(K)$, $\mu_\lambda\{\lambda\} = 0$, and $p(\lambda) = \int \bar{z} p\, d\mu_\lambda$ for each polynomial p.*

Therefore,

$$|p(\lambda)| = \left|\int \bar{z} p\, d\mu_\lambda\right| = \left|\int (\bar{z}p + r)\, d\mu_\lambda\right| \leq C_\lambda\|\bar{z}p + r\|_K.$$

Moreover, there exists $\delta = \delta(\lambda) > 0$ such that $\overline{O(\lambda, \delta)} \subset U$ and

$$|p(w)| \leq C_\lambda\|\bar{z}p + r\|_K$$

for each point $w \in O(\lambda, \delta)$. Hence, for each compact set $B \subset U$, there exists a constant $C > 0$ so that

$$|p(w)| \leq C\|\bar{z}p + r\|_K$$

for each point $w \in B$.

Suppose that λ is a nonpeak point for $R(K)$. Since U is an open set, from Theorem 4, one can choose a constant $\delta > 0$ such that $\overline{\Delta} = \overline{O(\lambda, \delta)} \subset U$. Next we will fix this δ.

Corollary 6. *Suppose $R(K) \neq C(K)$ and λ is a nonpeak point for $R(K)$. Let g be as in the main theorem and $|\bar{\partial} g(z)| \geq c > 0$ on $\overline{\Delta}$. Let h be a C^1 function on $\overline{\Delta}$. Suppose that $\bar{\partial} h/\bar{\partial} g$ is not in $R(K \cap \overline{\Delta})$ (notice that λ is a nonpeak point for $R(K \cap \overline{\Delta})$; hence, $R(K \cap \overline{\Delta}) \neq C(K \cap \overline{\Delta})$). Then h is not in the closure of $P(K)g + R(K)$.*

Proof. Suppose that h is in the closure of $P(K)g + R(K)$. Then there exists a sequence of polynomials $\{p_n\}$ and there exists a sequence of rational functions with poles off K such that $p_n g + r_n \to h$ uniformly on K. It follows from the facts after Theorem 5 that p_n uniformly converges to an analytic function p on $\overline{\Delta}$. Therefore, $\{r_n\}$ uniformly converges to r on $K \cap \overline{\Delta}$ and $h = pg + r$ on $K \cap \overline{\Delta}$. Since $h - pg$ is C^1 on $\overline{\Delta}$, we can extend r into a C^1 function on $\overline{\Delta}$ and

$$\bar{\partial} h = p\bar{\partial} g + \bar{\partial} r.$$

Using Lemma 1, we see that $\bar{\partial} r = 0$ on the set of nonpeak points for $R(K \cap \overline{\Delta})$. Hence, $\bar{\partial} h/\bar{\partial} g \in R(K \cap \overline{\Delta})$. This contradicts our assumption.

Corollary 7. *Suppose that* $R(K) \neq C(K)$. *Then the function* $\bar{z}^2$ *is not in the closure of* $P(K)\bar{z} + R(K)$.

Corollary 7 partially answers the question asked by Thomson at the end of his paper [T2].

Proof of our main theorem. Choose $\delta_0 > 0$ such that g is C^1 on $\overline{O(\lambda, \delta_0)}$.

Case 1. Suppose that $\bar{\partial} g = 0$ on the set of nonpeak points for $R(K \cap \overline{O(\lambda, \delta_0)})$. Using Lemma 1, we see that

$$g \in R(K \cap \overline{O(\lambda, \delta_0)}).$$

Hence, $P(K \cap \overline{O(\lambda, \delta_0)})g + R(K \cap \overline{O(\lambda, \delta_0)})$ is $R(K \cap \overline{O(\lambda, \delta_0)})$ which is not $C(K \cap \overline{O(\lambda, \delta_0)})$. Thus, $P(K)g + R(K)$ is not dense in $C(K)$.

Case 2. Suppose that there exists a nonpeak point λ_0 for $R(K \cap \overline{O(\lambda, \delta_0)})$ such that $\bar{\partial} g(\lambda_0) \neq 0$. Choose $\delta > 0$ such that $|\bar{\partial} g(z)| \geq c > 0$ on $\bar{\Delta} = \overline{O(\lambda_0, \delta)}$. Let ϕ be a smooth function with support in $\bar{\Delta}$, and let h be $T_\phi g$. Then $\bar{\partial} h = \phi \bar{\partial} g$; hence, there are a lot of smooth functions h such that $\bar{\partial} h / \bar{\partial} g \notin R(K \cap \bar{\Delta})$. From Corollary 6, one sees that the closure of $P(K)g + R(K)$ is not equal to $C(K)$.

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