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A RIM-METRIZABLE CONTINUUM

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ABSTRACT. A locally connected rim-metrizable continuum is constructed which admits a continuous mapping onto a non rim-metrizable space.

1. INTRODUCTION

We consider Hausdorff spaces and continuous mappings only. By a *continuum* we mean a compact and connected space. An *arc* is a linearly ordered continuum. It is well known that each separable arc is homeomorphic to $[0, 1]$. Recall that a space Y is said to be *scattered* if each nonempty subset of Y has an isolated point. We shall say that a space X is *rim-metrizable* (resp. *rim-countable* or *rim-scattered*) if X has a basis $\mathscr{B}$ of open sets such that $\mathrm{bd}(U)$ is metrizable (resp. countable or scattered) for each $U \in \mathscr{B}$. It is well known that each compact and countable space is both scattered and metrizable. Hence, each rim-countable compact space is rim-metrizable and rim-scattered.

Of course, each 0-dimensional compact space is rim-metrizable. Therefore, it is interesting to investigate compact rim-metrizable spaces which are not 0-dimensional only. Then it is natural to restrict attention to continua. That restriction is not strong enough yet, and the most interesting problems arise when locally connected rim-metrizable continua are studied.

Our aim is to construct a continuum X whose existence proves the following theorem:

Theorem 1. *The continuous image of a locally connected and rim-metrizable continuum need not be a rim-metrizable space.*

Theorem 1 provides the negative answer to a 1987 question of E. D. Tymchatyn. The desired continuum X is constructed as the inverse limit of an ω_1-long transfinite inverse sequence of copies of the square $[0, 1]^2$ with carefully chosen bonding maps. The construction of X and proofs of its properties are given in Section 3, while Section 2 contains auxiliary results. Recall here

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that [11] contains a rather simple example of a rim-metrizable continuum which is not locally connected and can be mapped onto a non-rim-metrizable space.

In 1967 S. Mardešič proved that each space which is a continuous image of an arc is rim-metrizable ([6], see also [7] and [5] for stronger results). Simple examples show that there exist locally connected rim-metrizable continua which are not continuous images of arcs. Rather complicated methods were employed in [8] in order to get a locally connected and rim-countable (whence: rim-metrizable) continuum which is the continuous image of no arc. Many nice results about rim-metrizable spaces were obtained in [9], [10] and [11]. In particular, the following Theorem 2 is related to our Theorem 1:

Theorem 2 [11, Theorem 3.5]. *If Y is a locally connected continuum which is the image of a rim-metrizable continuum under a pseudo-confluent mapping, then Y is rim-metrizable.*

Theorem 3 [11, Theorem 2.8]. *A product $X \times Y$ of compact spaces X and Y is rim-metrizable if and only if one of the following conditions is satisfied*:

(a) *both X and Y are metrizable*;
(b) *both X and Y are* 0-*dimensional*;
(c) *one of X and Y is rim-metrizable and the other is metrizable and zero-dimensional.*

Since rim-metrizability is a hereditary property, Theorem 3 yields the following Lemma 1 which will be needed later.

Lemma 1. *If a space Z contains a subset homeomorphic to the product $X \times [0, 1]$ of a non-metrizable compact space X and $[0, 1]$, then Z is not rim-metrizable.*

The reader is referred to [2] for general and well-known facts on inverse systems and their limits. More special properties of inverse limit spaces will be provided with appropriate references.

2. Auxiliary constructions

Let C be a copy of the Cantor set in the open interval $]0, 1[$. We shall consider C with its linear ordering inherited from $]0, 1[$.

Let Z be a subset of $[0, 1]^2$ such that Z is homeomorphic to $C \times [0, 1]$. Let $B = (\{0, 1\} \times [0, 1]) \cup ([0, 1] \times \{0, 1\})$ denote the boundary of the square $[0, 1]^2$. We shall say that Z is *well-placed* if $K \cap B$ consists of exactly two points which are the end points of K, for each component K of Z.

Now, let Z be a well-placed copy of $C \times [0, 1]$ in $[0, 1]^2$. Then $[0, 1]^2 - K$ has exactly two components, for each component K of Z. Let $h: Z \to C \times [0, 1]$ be a homomorphism and, for each component K of Z, let $c_K \in C$ be such that $h(K) = \{c_K\} \times [0, 1]$. We shall say that h is a *placement homeomorphism* if $h(Z \cap M) = \{c \in C : c < c_K\} \times [0, 1]$ or $h(Z \cap M) = \{c \in C : c > c_K\} \times [0, 1]$ for each component K of Z and each component M of $[0, 1]^2 - K$.

We omit proofs of the following two lemmas. Lemma 2 is quite trivial, and proofs of results similar to Lemma 3 can be found in [3].

Lemma 2. *If Z is a well-placed copy of $C \times [0, 1]$ in $[0, 1]^2$, then there exists a placement homeomorphism $h: Z \to C \times [0, 1]$.*

Lemma 3. *If* Z *is a well-placed copy of* $C \times [0, 1]$ *in* $[0, 1]^2$ *and* $h: Z \to C \times [0, 1]$ *is a placement homeomorphism, then* h *can be extended to a homeomorphism* $H: [0, 1]^2 \to [0, 1]^2$.

Let A be a subset of cardinality $\aleph_1$ of $]0, 1[$ such that the complement of A is dense in $[0, 1]$ (the latter assumption is going to simplify some arguments in the next section). Let $\{a_\alpha : \alpha < \omega_1\}$ be an enumeration of A.

Let $L = ([0, 1] \times \{0\}) \cup (A \times [0, 1])$ and order L lexicographically; i.e., let $\langle x, y\rangle < \langle x', y'\rangle$ if either $x < x'$ or $x = x'$ and $y < y'$. Then $<$ is a linear ordering on L. We consider L with its order topology introduced by the subbasis of intervals of the form $\{u : u < v\}$ or $\{u : v < u\}$, $v \in L - \{\langle 0, 0\rangle, \langle 1, 0\rangle\}$, for open sets. Then L is an ordered continuum ($\equiv$ a Hausdorff arc) which is nonseparable and, hence, nonmetrizable.

Define $r: L \to [0, 1]$ by $r(\langle x, y\rangle) = x$. Then r is a continuous onto map and $r^{-1}(x)$ is a nondegenerate closed sub-arc of L for each $x \in A$. For every $\alpha < \omega_1$ let $I_\alpha = r^{-1}(a_\alpha)$. Hence, each I_α is a copy of $[0, 1]$ and $I_\alpha \cap I_{\alpha'} = \varnothing$ if $\alpha \neq \alpha'$.

For $\alpha < \omega_1$, let $\mathcal{G}_\alpha$ denote the decomposition of L into the arcs I_β, $\alpha \leq \beta < \omega_1$, and points. Since each decomposition of an arc into sub-arcs and points is upper semicontinuous, the quotient space $L_\alpha = L/\mathcal{G}_\alpha$ is a Hausdorff arc again. Let $r_\alpha: L \to L_\alpha$ denote the quotient map.

Observe that $L_0 = [0, 1]$ and $r_0 = r$. Also, each arc L_α, $\alpha < \omega_1$, is separable and therefore homeomorphic to $[0, 1]$.

Now, suppose that $\alpha \leq \beta \leq \gamma < \omega_1$. Then $\mathcal{G}_\beta$ refines $\mathcal{G}_\alpha$. Hence, there is the unique $r_\alpha^\beta: L_\beta \to L_\alpha$ such that $r_\alpha = r_\alpha^\beta \circ r_\beta$. Clearly, r_α^β is continuous, monotone, and onto. Moreover, $r_\alpha^\gamma = r_\alpha^\beta \circ r_\beta^\gamma$. Thus, we obtain an inverse system $\mathcal{R} = (L_\alpha, r_\alpha^\beta, \alpha \leq \beta < \omega_1)$ of metrizable arcs L_α with monotone onto bounding maps r_α^β.

Note that L is canonically homeomorphic to lim inv $\mathcal{R}$. Furthermore, if λ is a limit ordinal number with $\lambda < \omega_1$, then $\mathcal{R}_\lambda = (L_\alpha, r_\alpha^\beta, \alpha \leq \beta < \lambda)$ is an inverse system such that L_λ is canonically homeomorphic to lim inv $\mathcal{R}_\lambda$. We identify L with lim inv $\mathcal{R}$ and each L_λ with lim inv $\mathcal{R}_\lambda$.

It is convenient to introduce the following notation. For each $\alpha < \omega_1$ let $s_\alpha = \mathrm{id}_C \times r_\alpha$ denote the product map of $C \times L$ onto $C \times L_\alpha$, i.e., $s_\alpha(c, u) = (c, r_\alpha(u))$ for all $c \in C$ and $u \in L$. Furthermore, for every $\alpha \leq \beta < \omega_1$, let $s_\alpha^\beta = \mathrm{id}_C \times r_\alpha^\beta: C \times L_\beta \to C \times L_\alpha$. Obviously, $\mathcal{S} = (C \times L_\alpha, s_\alpha^\beta, \alpha \leq \beta < \omega_1)$ is an inverse system such that lim inv $\mathcal{S} = C \times L$.

3. The main result

We are going to apply transfinite induction to construct spaces X_α and Z_α, $\alpha < \omega_1$, and mappings $t_\alpha^\beta: X_\beta \to X_\alpha$, $\alpha \leq \beta < \omega_1$, and $h_\alpha: C \times L_\alpha \to Z_\alpha$, $\alpha < \omega_1$, such that the following properties (1)–(6) are satisfied for all $0 \leq \alpha \leq \beta \leq \gamma < \omega_1$:

(1) $X_\alpha = [0, 1]^2$;
(2) $Z_\alpha \subset X_\alpha$ and $Z_\alpha = C \times [0, 1]$;
(3) h_α is a homeomorphism of $C \times L_\alpha$ onto $Z_\alpha = C \times [0, 1]$ such that the first coordinate of each point $h_\alpha(c, u)$ is c;
(4) $t_\alpha^\beta(Z_\beta) = Z_\alpha$ and $(t_\alpha^\beta|_{Z_\beta}) \circ h_\beta = h_\alpha \circ s_\alpha^\beta$;

(5) $t_\alpha^\beta(X_\beta - Z_\beta) = X_\alpha - Z_\alpha$ and $t_\alpha^\beta|_{X_\beta - Z_\beta} : X_\beta - Z_\beta \to X_\alpha - Z_\alpha$ is a homeomorphism;

(6) $t_\alpha^\alpha = \mathrm{id}_{X_\alpha}$ and $t_\alpha^\gamma = t_\alpha^\beta \circ t_\beta^\gamma$.

Let $X_0 = [0, 1]^2$, $Z_0 = C \times [0, 1] = C \times L_0$, and $h_0 = \mathrm{id}_{Z_0}$.

Suppose that for some ordinal number δ with $0 < \delta < \omega_1$, the required spaces X_α and Z_α, $\alpha < \delta$, and mappings t_α^β, $\alpha \leq \beta < \delta$, and h_α, $\alpha < \omega_1$, are already constructed.

First, consider the case when $\delta = \varepsilon + 1$. Let $X_\delta = [0, 1]^2$ and $Z_\delta = C \times [0, 1] \subset X_\delta$. Let $\mathscr{F}$ denote the decomposition of X_δ into the arcs $\{c\} \times [\frac{1}{3}, \frac{2}{3}]$, $c \in C$, and points. Also, let $f\colon X_\delta \to X_\delta/_{\mathscr{F}}$ denote the quotient map. Since $X_\delta/_{\mathscr{F}}$ is homeomorphic to the square, we may let $i\colon X_\delta/_{\mathscr{F}} \to [0, 1]^2$ be a homeomorphism. Clearly, $i(f(Z_\delta))$ is a well-placed copy of $C \times [0, 1]$.

Recall that there is exactly one point in L_ε whose pre-image under r_ε^δ is nondegenerate, and this point is not an end point of L_ε. Furthermore, $s_\varepsilon^\delta = \mathrm{id}_C \times r_\varepsilon^\delta$. Therefore, there exist homeomorphisms $h_\delta\colon C \times L_\delta \to Z_\delta$ and $j\colon i(f(Z_\delta)) \to Z_\varepsilon$ such that the first coordinate of each point $h_\delta(c, u)$ is c and $h_\varepsilon \circ s_\varepsilon^\delta = j \circ i \circ f \circ h_\delta$. Clearly, j must be a placement homeomorphism. By Lemma 3, there exists a homeomorphism $J\colon i(f(X_\delta)) = [0, 1]^2 \to X_\varepsilon = [0, 1]^2$ which extends j. It suffices to let $t_\varepsilon^\delta = J \circ i \circ f$ and $t_\alpha^\delta = t_\alpha^\varepsilon \circ t_\varepsilon^\delta$ for each $\alpha \leq \varepsilon$.

Now, consider the case when δ is a limit ordinal number, $0 < \delta < \omega_1$. By the inductive assumptions (6) and (4), $(X_\alpha, t_\alpha^\beta, \alpha \leq \beta < \delta)$ and $(Z_\alpha, t_\alpha^\beta|_{Z_\beta}, \alpha \leq \beta < \delta)$ are well-defined inverse systems. Let X_δ' and Z_δ' denote their inverse limits, respectively. Then $Z_\delta' \subset X_\delta'$. Also, let $\pi_\alpha\colon X_\delta' \to X_\alpha$, $\alpha < \delta$, denote the natural projections and let $h_\delta'\colon C \times L_\delta \to Z_\delta$ be the homeomorphism induced by the homeomorphisms h_α, $\alpha < \delta$.

For any sequence $(\varepsilon_n)_{n=1}^\infty$ of ordinals which increases to δ, there exists the natural homeomorphism of X_δ' onto the inverse limit space $\lim \mathrm{inv}(X_{\varepsilon_n}, t_{\varepsilon_n}^{\varepsilon_{n+1}})$ of the inverse sequence $(X_{\varepsilon_n}, t_{\varepsilon_n}^{\varepsilon_{n+1}})$. Since the spaces X_{ε_n} are copies of the square and all the maps $t_{\varepsilon_n}^{\varepsilon_{n+1}}$ are monotone, X_δ' is homeomorphic to the square (see [1, Theorem 4 and the Corollary following it on p. 482]). Let $i\colon X_\delta' \to [0, 1]^2$ be a homeomorphism. We already know that Z_δ' is homeomorphic to $C \times L_\delta$ and so to $C \times [0, 1]$. A simple direct proof shows that $i(Z_\delta')$ is a well-placed copy of $C \times [0, 1]$. Hence there exists a homeomorphism $j\colon [0, 1]^2 \to [0, 1]^2$ such that $j(i(Z_\delta')) = C \times [0, 1]$ and the first coordinate of $j \circ i \circ h_\delta'(c, u)$ is c for each $(c, u) \in C \times L_\delta$.

It suffices to let $X_\delta = j(i(X_\delta')) = [0, 1]^2$, $Z_\delta = j(i(Z_\delta')) = C \times [0, 1]$, $h_\delta = j \circ i \circ h_\delta'\colon C \times L_\delta \to Z_\delta$, and $t_\alpha^\delta = \pi_\alpha \circ i^{-1} \circ j^{-1} : X_\delta \to X_\alpha$ for each $\alpha < \delta$. This concludes the inductive construction.

By (6), $(X_\alpha, t_\alpha^\beta, \alpha \leq \beta < \omega_1)$ is an inverse system. We let X denote its inverse limit. Then X is a continuum as the inverse limit of continua. Since all the factor spaces X_α are locally connected continua and all the bonding maps t_α^β are monotone and onto, X is a locally connected continuum (see, e.g., [4]). Thus we have the following properties of X:

Claim 1. X is a locally connected continuum.

Now, let us prove two more special properties of X.

Claim 2. X is rim-metrizable.

Let $t_\alpha: X \to X_\alpha$, $\alpha < \omega_1$, denote the projections. If $\mathscr{B}_\alpha$ is a basis of X_α, for $\alpha < \omega_1$, then the collection $\mathscr{B}$ of all sets $t_\alpha^{-1}(U)$, $U \in \mathscr{B}_\alpha$, $\alpha < \omega_1$, is a basis of X. For each α, we are going to find a basis $\mathscr{B}_\alpha$ of X_α such that if $U \in \mathscr{B}_\alpha$, then $t_\alpha^{-1}(U)$ has metrizable boundary. Then $\mathscr{B}$ as above will be a basis of X which consists of open sets with metrizable boundaries.

Let $\alpha < \omega_1$. Let $P_\alpha = \{x \in X_\alpha : t_\alpha^{-1}(x) \text{ is nondegenerate}\}$. If $x \in P_\alpha$, then $x \in Z_\alpha$ and $s_\alpha^{-1}(h_\alpha^{-1}(x))$ is nondegenerate, because $s_\alpha^{-1}(h_\alpha^{-1}(x)) = I_\beta$ for some β such that $\alpha \leq \beta < \omega_1$. It follows that $h_\alpha^{-1}(P_\alpha)$ is a subset of $C \times L_\alpha$ which is contained in $C \times M_\alpha$, where M_α is the set of all points $(r_0^\alpha)^{-1}(a_\beta)$, $\alpha \leq \beta < \omega_1$. Since $[0, 1] - A$ is dense in $[0, 1]$, the set M_α is 0-dimensional. By Lemma 3, we may assume that $P_\alpha \subset C \times N_\alpha$ for some 0-dimensional subset N_α of $[0, 1]$. Now, let $\mathscr{B}_\alpha$ be a countable basis of $X_\alpha = [0, 1]^2$ such that $\text{bd}(U) \cap (C \times N_\alpha) = \varnothing$ for each $U \in \mathscr{B}_\alpha$. Let $U \in \mathscr{B}_\alpha$, and observe that $\text{bd}(t_\alpha^{-1}(U)) = t_\alpha^{-1}(\text{bd}(U))$ and t_α is one-to-one on the set $\text{bd}(t_\alpha^{-1}(U))$. Hence, t_α maps $\text{bd}(t_\alpha^{-1}(U))$ homeomorphically onto $\text{bd}(U) \subset [0, 1]^2$, and so $\text{bd}(t_\alpha^{-1}(U))$ is metrizable. This completes the proof of Claim 2.

Claim 3. X can be mapped onto a space which is not rim-metrizable.

Let

$$Z = \lim\operatorname{inv}(Z_\alpha, t_\alpha^\beta|_{Z_\beta}, \alpha \leq \beta < \omega_1) = \bigcap_{\alpha<\omega_1} t_\alpha^{-1}(Z_\alpha) \subset X.$$

The homeomorphisms h_α, $\alpha < \omega_1$, induce the homeomorphism $h: C \times L \to Z$.

Let f be any mapping of the Cantor set C onto $[0, 1]$. Define $g: C \times L \to [0, 1] \times L$ by $g(c, u) = (f(c), u)$.

Let $\mathscr{G}$ be the decomposition of X into the sets $h(g^{-1}(w))$, $w \in [0, 1] \times L$, and points. Then $\mathscr{G}$ is upper semicontinuous, whence the quotient space $X/\mathscr{G}$ is Hausdorff. Observe that $X/\mathscr{G}$ contains a subset homeomorphic to $[0, 1] \times L$. Since L is non-metrizable, $X/\mathscr{G}$ is not rim-metrizable, by Lemma 1.

4. Remarks

1. The spaces X and Z were constructed by means of monotone mappings and inverse limits. However, the decomposition $\mathscr{G}$ in the proof of Claim 3 is not monotone. Furthermore, the construction cannot be modified to provide a monotone decomposition $\mathscr{G}'$ with all the required properties. Indeed, by Theorem 2, a monotone image of a locally connected rim-metrizable continuum is rim-metrizable again.

2. Let M be a subset of X which consists of all points y such that either $y = f_0^{-1}(x)$ for some $x \in]0, 1[-(C \times [0, 1])$ or $y = h(c, u)$ for some $c \in C$ and $u \in L$ such that $u \in \bigcup_{\alpha<\omega_1} I_\alpha = r^{-1}(A)$ and u is not an end point of any I_α. One can show that M is a Hausdorff 2-manifold, i.e., each point of M has an open neighborhood homeomorphic to $]0, 1[^2$. Also, M is a $T_{3\frac{1}{2}}$-space which is not normal.

3. The space X we constructed above is 2-dimensional. It contains a closed subspace Y such that Y is a locally connected curve which has a basis of open sets with metrizable 0-dimensional boundaries and yet $Z \subset Y$, whence continuous images of Y need not be rim-metrizable.

In fact, let S be a copy of the Sierpiński universal plane curve such that $C \times [0, 1] \subset S \subset [0, 1]^2 = X_0$ and let $Y = t_0^{-1}(S)$. By [12], it easily follows that $(t_0^{\alpha})^{-1}(S)$ is a copy of S contained in X_α, $\alpha < \omega_1$. A modification of the proof of Claim 2 provides a basis of Y as needed.

4. Continuous images of rim-countable continua and locally connected rim-scattered continua were considered in [10], where it was proved that they can contain no subset homeomorphic to the product of a nonmetric compact space and a perfect set. It would be interesting to know if continuous images of locally connected rim-countable continua must be rim-metrizable. Also, it is unknown if one can find a perfectly normal space which is not rim-metrizable and is the continuous image of a locally connected rim-metrizable continuum. Some other questions concerning rim-properties of continua can be found in [8, p. 85]. In particular, it is asked there (see also [7]) if a locally connected rim-scattered continuum must be rim-countable.

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