PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 1, January 1995

ON p-HYPONORMAL CONTRACTIONS

B. P. DUGGAL

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The contraction A on a Hilbert space H is said to be p-hyponormal, $0 < p < 1$, if $(A^*A)^p \geq (AA^*)^p$. Let A be an invertible p-hyponormal contraction. It is shown that A has $C_{.0}$ completely nonunitary part. Now let H be separable. If A is pure and the defect operator $D_A = (1 - A^*A)^{1/2}$ is of Hilbert-Schmidt class, then $A \in C_{10}$. Let B^* be a contraction such that B^* has $C_{.0}$ completely nonunitary part, D_{B^*} is of Hilbert-Schmidt class, and B^* satisfies the property that if the restriction of B^* to an invariant subspace is normal, then the subspace reduces B^*. It is shown that if $AX = XB$ for some quasi-affinity X, then A and B are unitarily equivalent normal contractions.

1. INTRODUCTION

Let H be a Hilbert space, and let $B(H)$ denote the algebra of operators, i.e., bounded linear transformations, on H into itself. An operator $A \in B(H)$ is said to be hyponormal if $A^*A \geq AA^*$. The class of semihyponormal or $\frac{1}{2}$-hyponormal, operators was introduced by Xia [12] by weakening the hyponormality condition $A^*A \geq AA^*$ to $(A^*A)^{1/2} \geq (AA^*)^{1/2}$. More generally, $A \in B(H)$ is said to be p-hyponormal, $0 < p < 1$, if $(A^*A)^p \geq (AA^*)^p$. The class of p-hyponormal operators has been studied in a number of papers, mainly by Xia and Aluthge (see [1, 2, 12, 13] and some of the references in [2]). Although the class of p-hyponormal operators is independent of the class of hyponormal operators (there exist p-hyponormal operators which are not hyponormal [1, 2, 13]), it is nevertheless true that p-hyponormal operators share many properties with hyponormal operators (see [2] for some instances of this). In this note we consider invertible p-hyponormal contractions A, and show that (just as for hyponormal contractions) the completely nonunitary part of A is of class $C_{.0}$. Also it will be shown that (in common with hyponormal contractions) if the Hilbert space H is separable, A is pure, and the defect operator D_A is of Hilbert-Schmidt class, then $A \in C_{10}$. We also prove a Putnam-Fuglede theorem type commutativity result for p-hyponormal contractions.

Received by the editors November 5, 1991.
1991 *Mathematics Subject Classification*. Primary 47B10, 47B20, 47A10.
Key words and phrases. p-hyponormal contraction, Hilbert-Schmidt class, quasi-similar, C_0-contraction.

2. Notation and terminology

The closure of the range and the orthogonal complement of the kernel of an $X \in B(H)$ will be denoted by $\overline{\operatorname{ran} X}$ and $\ker^{\perp} X$, respectively. An $X \in B(H)$ is said to be a quasi-affinity if both X and X^* have dense range, and $A, B \in B(H)$ are said to be quasi-similar if there exist quasi-affinities $X, Y \in B(H)$ such that $AX = XB$ and $BY = YA$. We shall denote the (open) unit disc (in the complex plane) by D, and the boundary of D will be denoted by ∂D. The spectrum, the joint point spectrum, and the point spectrum of an $A \in B(H)$ will be denoted by $\sigma(A)$, $\sigma_{jp}(A)$, and $\sigma_p(A)$. (The point $\lambda = re^{i\theta}$, $r \geq 0$, is said to be in the joint point spectrum of $A \in B(H)$, where A has polar decomposition $A = V|A|$, if there exist a common eigenvector $f \neq 0$ of V and $|A|$ such that $Vf = e^{i\theta} f$ and $|A|f = rf$.)

The contraction A is said to be c.n.u. (= completely nonunitary) if there exists no nontrivial reducing subspace H_0 of H such that $A|H_0$ is unitary. We say that the contraction A is of class $C_{.0}$, $A \in C_{.0}$, if $A^{*n} \to 0$ strongly as $n \to \infty$ and that $A \in C_{1.}$ if $\inf_n \|A^n x\| > 0$ for all nonzero $x \in H$. The class $C_{\alpha\beta}$, for $\alpha, \beta = 0, 1$, is defined by $C_{\alpha\beta} = C_{\alpha.} \cap C_{.\beta}$. The c.n.u. contraction A belongs to the class C_0 if there exists an inner function $\varnothing$ such that $\varnothing(A) = 0$. Recall that if $A \in C_0$, then amongst all inner functions $\varnothing$ such that $\varnothing(A) = 0$ there is a minimal one (i.e., one which is a divisor, in the Hardy space H^∞, of all others) called the minimal function of A. The defect operator D_A of the contraction A is defined by $D_A = (1 - A^*A)^{1/2}$; D_A is said to be of the Hilbert-Schmidt class if trace $D_A^2 < \infty$.

3. Results

Throughout the following A will denote an invertible p-hyponormal contraction $(0 < p < 1)$ with polar decomposition

$$A = V|A|, \qquad V \text{ unitary and } |A| > 0.$$

The contraction $\widehat{A}$ will be defined by

$$\widehat{A} = |A|^{1/2} V |A|^{1/2}, \tag{1}$$

and it will be assumed that $\widehat{A}$ has polar decomposition

$$\widehat{A} = \widehat{V}|\widehat{A}|.$$

The contraction W will be defined by

$$W = |\widehat{A}|^{1/2} \widehat{V} |\widehat{A}|^{1/2}. \tag{2}$$

Then A is similar to W; and if $0 < p < \frac{1}{2}$, then W is hyponormal [2, Corollary 3.3].

Theorem 1. *The c.n.u. part of A is of class $C_{.0}$.*

Proof. By Löwner's theorem [7] a p-hyponormal operator is q-hyponormal for $q \leq p$. Hence if $\frac{1}{2} \leq p < 1$ and A is p-hyponormal, then A is q-hyponormal for $0 < q < \frac{1}{2}$. Thus it is sufficient to prove the theorem for $0 < p < \frac{1}{2}$.

Suppose A is c.n.u. Let $(0 \neq) y \in H$, and let $\{y_n\}$ be the sequence defined by $y_n = A^{*n}y$. Then $\{\|y_n\|\}$ is a monotonic decreasing bounded numerical sequence which converges to its greatest lower bound p_y (say). We have two possibilities: either $p_y = 0$ or $p_y > 0$. If $p_y = 0$ for all $y \in H$, then $A \in C_{.0}$ and we are done; if $p_y > 0$, then let M be the subspace

$$M = \{y \in H : 0 < p_y \leq \|A^{*n}y\|,\ n = 0, 1, 2, \dots\}.$$

Then there exists a co-isometry U and a quasi-affinity X such that

$$AX = XU$$

(this follows from the proof of [8, Proposition II.5.3]). Define the quasi-affinity S by $S = |\widehat{A}|^{-1/2}\widehat{V}^*|A|^{-1/2}V^*X$; then

$$WS = SU, \tag{3}$$

where W is hyponormal. By the Putnam-Fuglede theorem for hyponormal W and co-hyponormal U applied to (3) we conclude that W and U are unitarily equivalent unitary operators [10]. Since A is similar to W, A is similar to a unitary operator, and so $\sigma(A) \subseteq \partial D$.

Recall that

$$\|A\| = r_{\text{sp}}(A) \quad (= \text{spectral radius of } A)$$

[2, Theorem 3.1] and

$$\|A^{-1}\| = \frac{1}{\min\{|z| : z \in \sigma(A)\}}$$

[2, Theorem 3.12(3)]. Hence $\|A\| = \|A^{-1}\| = 1$. We have

$$\|Ax\|^k \leq \|x\|^k = \|x\|\,\|x\|^{k-1} = \|A^{-k}A^kx\|\,\|x\|^{k-1} \leq \|A^kx\|\,\|x\|^{k-1}$$

for given natural numbers $k \geq 2$ and all $x \in M$. Consequently, A is k-paranormal. (An operator T is said to be k-paranormal on H if $\|Tx\|^k \leq \|T^kx\|\,\|x\|^{k-1}$ for all $x \in H$.) Since a k-paranormal contraction similar to a unitary is unitary [3, Corollary 1], M reduces A and $A|M$ is unitary—a contradiction. Hence $A \in C_{.0}$.

Corollary 1. *If $AXB = X$ for some contraction B^* with $C_{.0}$ c.n.u. part and operator X, then $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B^*, and $A|\overline{\operatorname{ran} X}$ and $B^*|\ker^{\perp} X$ are unitarily equivalent unitary operators.*

Proof. Since A has $C_{.0}$ c.n.u. part, [4, Theorem 2(a)] applies.

It is immediate from Theorem 1 that the pure part of A (i.e., the completely p-hyponormal part of A) has representation of type

$$\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}.$$

Takahashi and Uchiyama [11, Theorem 4] have shown that a pure hyponormal contraction B (on a separable Hilbert space H) with the defect operator D_B in the Hilbert-Schmidt class (denoted in the sequel by C_2) if of class C_{10}. That this result holds for A such that $D_A \in C_2$ is the content of our next theorem. We shall assume henceforth that our Hilbert space H is separable.

Theorem 2. *If A is pure and $D_A \in C_2$, then $A \in C_{10}$.*

Proof. We start by considering the case $0 < p < \frac{1}{2}$. Then the contraction $\widehat{A}$ (of (1)) is $(p + \frac{1}{2})$-hyponormal and so semihyponormal ($= \frac{1}{2}$-hyponormal) by Löwner's theorem. This implies

$$\widehat{V}^*|\widehat{A}|\widehat{V} \geq |\widehat{A}| \geq \widehat{V}|\widehat{A}|\widehat{V}^*.$$

Set $T = V^*|A|V$ and $S = |A|$; then

$$|\widehat{A}|^2 = \widehat{A}^*\widehat{A} = |A|^{1/2}V^*|A|V|A|^{1/2} = S^{1/2}TS^{1/2} \geq S^{1/2+1/2+1} = S^2 = |A|^2$$

(see Furuta [6]). Hence

$$|\widehat{A}|^{1/2}\widehat{V}^*|\widehat{A}|\widehat{V}|\widehat{A}|^{1/2} \geq |\widehat{A}|^2 \geq |A|^2$$

and

$$(1 - A^*A) = 1 - |A|^2 \geq 1 - |\widehat{A}|^{1/2}\widehat{V}^*|\widehat{A}|\widehat{V}|\widehat{A}|^{1/2} = 1 - W^*W \quad (\geq 0).$$

Since $D_A \in C_2$, $D_W \in C_2$. The contraction W being hyponormal, and similar to A, is of class $C_{.0}$ (since A is, by Theorem 1). Hence W has direct sum decomposition $W_0 \oplus W_{10}$, where $W_0 \in C_0$ and $W_{10} \in C_{10}$ [11, Theorems 1 and 4]. Also, W_0 is normal, so that $\sigma(W_0) = \sigma_p(W_0) \subset D$ is countable [8, Theorem III.5.1].

Suppose A has a nontrivial C_{00} part. Then

$$A = \begin{pmatrix} A_0 & * \\ 0 & A_1 \end{pmatrix}, \quad \text{where } A_0 \in C_{00} \text{ and } A_1 \in C_{10}.$$

The similarity of A and W implies the existence of invertible operators $Y = [Y_{ij}]_{i,j=1}^2$ and $Z = [Z_{ij}]_{i,j=1}^2$ ($Z^{-1} = Y$) such that $AY = YW$ and $ZA = WZ$. Since $A_1Y_{21} = Y_{21}W_0$ and $Z_{21}A_0 = W_{10}Z_{21}$, $Y_{21} = 0 = Z_{21}$. Thus $A_0Y_{11} = Y_{11}W_0$ and $Z_{11}A_0 = W_0Z_{11}$, where Y_{11} and Z_{11} are injective. Consequently, A_0 and W_0 are quasi-similar C_0 contractions [9, Theorem 1] and so have the same spectrum [8]. Thus $\sigma(A_0) = \sigma_p(A_0) \subset D$ is countable.

Let m_{A_0} denote the minimal function of A_0. Then since A_1 has empty point spectrum, the point spectrum of A consists of the zeros of m_{A_0}. Since $\sigma_{jp}(A) = \sigma_p(A)$ (see [13; 2, Theorem 3.8]) and the eigenspaces of V reduce A (see [13; 2, Theorem 3.5]), $A = A_0 \oplus A_1$. We assert that A_0 is normal: this would then imply that A could not have a nontrivial C_{00} part and hence that $A \in C_{10}$ (completing thereby the proof for the case $0 < p < \frac{1}{2}$).

Let $B = V|A_0|^p$; then B is a hyponormal contraction with $\sigma(B) = \sigma_p(B) = \tau(\sigma(A_0)) = \tau(\sigma_p(A_0))$, where $\tau(re^{i\theta}) = r^p e^{i\theta}$ [2, Theorem 3.14]. Since $\sigma(B) = \sigma_p(B)$ is countable, $\sigma(B)$ has zero Lebesgue area measure, implying thereby that B is normal. Thus

$$0 = B^*B - BB^* = (A_0^*A_0)^p - (A_0A_0^*)^p,$$

i.e., A_0 is normal.

To complete the proof we note (as before) that if A is p-hyponormal, $\frac{1}{2} \leq p < 1$, then A is q-hyponormal for $q \leq p$. Hence the argument above applies.

We prove now a Putnam-Fuglede theorem for p-hyponormal contractions. It will be assumed in the following theorem that the contraction B^* satisfies the property: If the restriction of B^* to an invariant subspace is normal, then the subspace reduces B^*.

Theorem 3. *If $AX = XB$ for some quasi-affinity X and contraction B^*, with $C_{.0}$ c.n.u. part, satisfying $D_{B^*} \in C_2$, then A and B are unitarily equivalent normal contractions.*

Proof. Let S be the quasi-affinity $S = |\widehat{A}|^{-1/2}\widehat{V}^*|A|^{-1/2}V^*X$. Then, with W as defined in (2), $WS = SB$. Since W is hyponormal, W and B are unitarily equivalent normal contractions of type unitary $\bigoplus C_0$ (apply [5, Theorem 1′]). Since A is similar to W, $AY = YW$ for some invertible operator Y. We show that A has no pure part. For suppose that A has the direct sum decomposition $A = A_n \oplus A_c$, where A_n is normal and $A_c \in C_{.0}$ is pure. Decompose W by $W = W_u \oplus W_0$, where W_u is unitary and $W_0 \in C_0$ is normal, and let Y have the corresponding matrix representation $Y = [Y_{ij}]_{i,j=1}^2$. Then, since $A_c Y_{21} = Y_{21} W_u$, $Y_{21} = 0$. We have $A_c Y_{22} = Y_{22} W_0$ or $A_c(Y_{22}|\ker^\perp Y_{22}) = (Y_{22}|\ker^\perp Y_{22})(W_0^*|\ker^\perp Y_{22})^*$. Since Y_{22} has dense range and $W_0 \in C_0$ is normal, $A_c \in C_0$ and $\sigma(A_c) = \sigma_p(A_c) \subset D$ is countable. Since A_c is p-hyponormal, an argument similar to that used in the proof of Theorem 2 shows that A_c is normal—a contradiction. This completes the proof.

Recall that given a C_{10} contraction B^* there exists an isometry L and a quasi-affinity Z satisfying $LZ = ZB^*$. Hence if $AX = XB$, $B^* \in C_{10}$, for some quasi-affinity X, then $AXZ^* = XZ^*L^*$. Applying Theorem 3 it follows that A is unitarily equivalent to a unitary operator. Hence

$$\|Xx\| = \|A^n Xx\| = \|XB^n x\| \le \|X\|\,\|B^n x\| \to 0 \quad \text{as } n \to \infty$$

for all $x \in H$. We have:

Corollary 2. *There exists no quasi-affinity X such that $AX = XB$ for some C_{10} contraction B^*.*

Acknowledgment

It is my pleasure to thank Dr. Ariyadasa Aluthge for supplying me with a copy of [2].

References

1. A. Aluthge, *On p-hyponormal operator for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.
2. ———, *Properties of p-hyponormal operators*, unpublished dissertation, Vanderbilt University, Nashville, TN, August 1990.
3. B. P. Duggal, *Contractions with a unitary part*, J. London Math. Soc. **31** (1985), 131–136.
4. ———, *On intertwining operators*, Monatsh. Math. **106** (1988), 139–148.
5. ———, *A generalized commutativity theorem*, Z. Anal. Anwendungen **10** (1991), 265–274.
6. T. Furuta, *$A \ge B \ge 0$ assures $(B^r A^p B^r)^{1/q} \ge B^{(p+2r)/q}$ and $A^{(p+2)/q} \ge (A^r B^p A^r)^{1/q}$ for $r \ge 0$, $q \ge 1$, with $(1+2r)q \ge (p+2r)$*, Proc. Amer. Math. Soc. **101** (1981), 85–88.
7. K. Löwner, *Uber monotone matrix functionen*, Math. Z. **38** (1934), 177–216.
8. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.
9. ———, *On injections intertwining operators of class* C_0, Acta Sci. Math. (Szeged) **40** (1978), 163–167.
10. J. G. Stampfli and B. L. Wadhwa, *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25** (1976), 359–365.

11. K. Takahashi and M. Uchiyama, *Every* C_{00} *contraction with Hilbert-Schmidt defect operator is of class* C_0, J. Operator Theory **10** (1983), 331–335.

12. D. Xia, *On the non-normal operators—semihyponormal operators*, Sci. Sinica **23** (1980), 700–713.

13. ——, *Spectral theory of hyponormal operators*, Birkhäuser Verlag, Boston, 1983.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF LESOTHO, P.O. ROMA 180, LESOTHO, SOUTHERN AFRICA

Current address: Department of Mathematics, Sultan Qaboos University, College of Science, P.O. Box 36, Al Khod 123, Oman