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POSITIVE DEFINITE FUNCTIONS OF HOPF C^*-ALGEBRAS

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ABSTRACT. In this paper we study positive definite functions of Hopf C^*-algebras. First of all, we introduce Fourier transformation on Hopf C^*-algebras and use Fourier transform to characterize positive definite functions. Then we proceed to study smooth positive definite functions on Hopf C^*-algebras. A complete description of smooth positive definite functions is obtained. Also, a Bochner type result for smooth positive definite functions is proved.

1. PRELIMINARIES

In this section we first recall some of the terminology and results which we will need throughout this paper. Then we describe the contents of this paper. Let us start with the definitions of representations and the Peter-Weyl property for Hopf C^*-algebras.

Let (A, Φ, k, e) be a Hopf C^*-algebra with dense *-subalgebra $\mathscr{A}$. We say that it is involutive if $k^2 = I$. For the definition and more information about Hopf C^*-algebras, we refer to [W1] and [V]. We start with the definition of comodules and representations of Hopf C^*-algebras.

Let $\mathscr{A}$ be a coalgebra, M a linear space, and

$$\psi : M \to \mathscr{A} \otimes M$$

a linear map which satisfies $(e \otimes I)\psi(m) = m$ and $(I \otimes \psi)\psi(m) = (\Phi \otimes I)\psi(m)$ for all $m \in M$. The pair (M, ψ) is called a left $\mathscr{A}$-comodule, and ψ is said to be its structure map. A right $\mathscr{A}$-comodule can be defined similarly. For more information about comodules, we refer to [A].

Two left $\mathscr{A}$-comodules (M_1, ψ_1), (M_2, ψ_2) are said to be isomorphic if there is an isomorphic linear space map $\phi : M_1 \to M_2$ such that $(I \otimes \phi) \cdot \psi_1 = \psi_2 \cdot \phi$.

Let M be a left $\mathscr{A}$-comodule with structure map ψ. A subspace $M_1 \subset M$ is said to be left invariant if $\psi(M_1) \subset \mathscr{A} \otimes M_1$; we say that M is an irreducible left $\mathscr{A}$-comodule if M is the only nonzero left invariant subspace of M.

Let (A, Φ, k, e) be a Hopf C^*-algebra, $\mathscr{A}$ a dense *-subalgebra of A, and V a finite-dimensional left $\mathscr{A}$-comodule with structure map $\psi : V \to \mathscr{A} \otimes V$.

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If $\{e_i\}_{i=1}^n$ is a basis of V and

$$\psi(e_i) = \sum_{k=1}^{n} a_{ik} \otimes e_k ,$$

the matrix (a_{ij}) is called the coefficient matrix of ψ with respect to $\{e_i\}_{i=1}^n$, and the comodule property implies that $\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$.

Now suppose that V is a left $\mathscr{A}$-comodule, which is also a Hilbert space with inner product $(\cdot, \cdot)\colon V \times V \to C$ and structure map $L\colon V \to \mathscr{A} \otimes V$. We can extend $(\cdot, \cdot)$ to $\langle \cdot, \cdot \rangle\colon (\mathscr{A} \otimes V) \times (\mathscr{A} \otimes V) \to \mathscr{A}$ as

$$\langle a \otimes x, b \otimes y \rangle = ab^*(x, y) \quad \forall a, b \in \mathscr{A}, x, y \in V.$$

A left unitary representation π of A on a Hilbert space H is a comodule map from H into $\mathscr{A} \otimes H$ such that $\langle \pi(x), \pi(y) \rangle = \langle x, y \rangle$, $\forall x, y \in H$. Two unitary representations π_1 and π_2 of A on H_1, H_2 are said to be unitary equivalent if there exists a unitary operator $U\colon H_1 \to H_2$ such that $(I \otimes U) \cdot \pi_1 = \pi_2 \cdot U$.

Let $\sigma \in A^*$ be a positive linear functional. We say that σ is a left Haar measure on A if

$$x^* \cdot \sigma = \langle x^*, I \rangle \cdot \sigma \quad \forall x^* \in A^*.$$

Similarly, σ is called a right Haar measure on A if $\sigma \cdot x^* = \langle x^*, I \rangle \cdot \sigma \ \forall x^* \in A^*$.

Now if σ is the normalized Haar measure on A, for $x \in \mathscr{A}$, then we can view it as an element of A^* as follows: $\forall y \in A$, $x(y) = \sigma(xy)$; we denote the completion of $\mathscr{A}$ on A^* as $L^1(A)$. For $x \in \mathscr{A}$, we denote $\|x\|_2 = (\sigma(xx^*))^{1/2}$ and the Hilbert completion of $\mathscr{A}$ under this inner product as $L^2(A)$.

Theorem 1.1. *Let (A, Φ, k, e) be an involutive Hopf C^*-algebra with dense *-subalgebra $\mathscr{A}$, and let V_π be an irreducible finite-dimensional left unitary $\mathscr{A}$-comodule with structure map π and* $\dim V_\pi = n$. *Let $\{e_1, \dots, e_t\}$ be an orthonormal set in V_π. Then we have*

(1) $\sigma(\langle Le_j, I \otimes e_k \rangle k(\langle Le_l, I \otimes e_m \rangle)) = (1/d)\delta_{jl}\delta_{km}$ *for* $j, k, l, m \in \{1, 2, \dots, t\}$.

(2) *If $\{e_1, \dots, e_n\}$ is an orthonormal basis of V_π and $\{a_{ij}^\pi : i, j = 1, \dots, n\}$ is the matrix coefficient of π with respect to the basis, then the set of elements $n^{1/2}a_{ij}^\pi \in A$, $i, j \in \{1, 2, \dots, n\}$, is an orthonormal set in $L^2(A, \sigma)$. Furthermore, if π_1 is a finite-dimensional irreducible left unitary representation of A on V_{π_1}, $\{a_{ij}^{\pi_1} : i, j = 1, \dots, \dim V_{\pi_1}\}$ is the matrix coefficient of π_1 with respect to an orthonormal basis of V_{π_1}, and π, π_1 are not unitary equivalent, then*

$$\sigma(a_{jk}^\pi (a_{lm}^{\pi_1})^*) = (1/n)\delta_{jl}\delta_{km}\delta_{\pi, \pi_1}$$

for all $j, k \in \{1, 2, \dots, n\}$, $l, m \in \{1, 2, \dots, \dim V_{\pi_1}\}$.

For a proof of this result, we refer to [Q1]; it is closely related to the orthogonal relations proved by Woronowicz in [W1].

Definition 1.2. Let Σ be the set of equivalence classes of all irreducible finite-dimensional unitary representations of the Hopf C^*-algebra (A, Φ, k, e). For $\pi \in \Sigma$, let $\{a_{ij}^\pi : 1 \le i, j \le d_\pi\}$ be the matrix coefficient of π on H_π with respect to a fixed orthonormal basis $\{e_i^\pi\}$ of H_π. Let A_R be the *-subalgebra of A generated by the set $\{a_{ij}^\pi : 1 \le i, j \le d_\pi, \pi \in \Sigma\}$; it is called the

representation algebra of A. If A_R is dense in A and $\{a_{ij}^{\pi} : 1 \leq i, j \leq d_\pi, \pi \in \Sigma\}$ is a basis for A_R, then we say that A has the Peter-Weyl property.

Note that Definition 1.2 has its obvious motivation from the classical Peter-Weyl theorem; also from the work of Woronowicz [W2], we know that every compact matrix pseudogroup has the Peter-Weyl property.

Throughout this paper, in order to keep the presentation simple, we always assume that every Hopf C^*-algebra is involutive, because with slight modification our arguments also work for general Hopf C^*-algebras.

Let I be an index set. For any $i \in I$, H_i is a finite-dimensional Hilbert space and $B(H_i)$ are all bounded operators on H_i. For each $i \in I$, let $d_i = \dim H_i$; choose an orthonormal basis $e_1^i, \ldots, e_{d_i}^i$. For any $x = \Sigma_k a_k^i e_k^i \in H_i$, define

$$\phi_p(x) = \begin{cases} \left(\sum_k |a_k^i|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_k |a_k^i| & \text{if } p = \infty. \end{cases}$$

Then for each $x \in B(H_i)$, we have $||x||_{\phi_p} = \sup\{|\operatorname{tr}(xy)| : ||y||_{\phi_{p'}} \leq 1\}$, where $1/p + 1/p' = 1$.

Now we turn to the contents of this paper. In §2 we study the Fourier transform on Hopf C^*-algebras and some of its basic properties. In §3 we study the Fourier series in $L^2(A)$, which provides us with the motiviation to define the positive definite functions on Hopf C^*-algebras, which is the main object of this paper. In §4 we introduce positive definite functions on Hopf C^*-algebras and show that, as in the compact group case, positive definite functions can be characterized by its Fourier transform. In §5 we study smooth positive definite functions on Hopf C^*-algebras. First of all, we introduce an equivalence relation on smooth positive definite functions. Then under this equivalence relation, we give a complete characterization of smooth positive definite functions. Also, we determine the extreme points of the set of normalized smooth positive definite functions, and a Bochner-type result for smooth positive definite functions is also proved.

2. Fourier transformation

In this section we introduce Fourier transform on Hopf C^*-algebras. We expect that it will play an important role on the future study of harmonic analysis on Hopf C^*-algebras. We start by recalling a class of Banach spaces on which our study will be based. Let (A, Φ, k, e) be a Hopf C^*-algebra with the Peter-Weyl property. Let Σ denote the set of all equivalence classes of irreducible finite-dimensional left unitary representations of A. For a subset P of Σ, $\mathscr{L}(P)$ always denotes the space defined by $\prod_{\pi \in P} B(H_\pi)$, where H_π is the representation space of π. If $E = (E_\pi)$ is an element in $\mathscr{L}(P)$, define

$$||E||_p = \left(\sum_{\pi \in P} d_\pi ||E_\pi||_{\phi_p}^p\right)^{1/p}$$

for $1 \leq p < \infty$ and

$$||E||_\infty = \sup\{||E_\pi||_{\phi_\infty} : \pi \in P\}.$$

In the following, we let $C_{00}(\Sigma) = \{E \in \mathscr{L}(\Sigma)\colon \text{the set } \{\pi \in \Sigma\colon E_\pi \neq 0\} \text{ is finite}\}$.

Let $\{e_1^\pi, \dots, e_{d_\pi}^\pi\}$ be an orthonormal basis for H_π, and assume that $\pi(e_i^\pi) = \sum_{j=1}^{d_\pi} a_{ij}^\pi \otimes e_j^\pi$ and $\bar{\pi}$ is the conjugate representation of π. Then we have $\bar{\pi}(e_i) = \sum_j k(a_{ij}^\pi) \otimes e_j$.

For $x^* \in A^*$, define $\hat{x}^*(\pi) \in B(H_\pi)$ as $\langle \hat{x}^*(\pi)\xi, \eta\rangle = x^*(\langle \bar{\pi}\xi, I \otimes \eta\rangle)$ for $\xi, \eta \in H_\pi$. Clearly, $\hat{x}^*$ belongs to $\mathscr{L}(\Sigma)$.

Definition 2.1. $\hat{x}^*$ is called a Fourier-Stieltjes transform of x^*.

Before we prove the next result, we need the following lemma.

Lemma 2.2. *Let (A, Φ, k, e) be an involutive Hopf C^*-algebra. For $x^*, y^* \in A^*$, define a linear functional $x^* * y^*$ on A by $\langle a, x^* * y^*\rangle = \langle \Phi(a), x^* \otimes y^*\rangle$. Then $(A^*, *)$ is a Banach algebra; if we define $k^*(x^*)(a) = x^*(k(a))$, then k^* is an involution for $(A^*, *)$.*

For a proof of Lemma 2.2, we refer to [V].

Theorem 2.3. *For each $\pi \in \Sigma$, $x^* \in A^*$, $\hat{x}^*$ defined as above, we have*

(1) *for each $\pi \in \Sigma$, the map $x^* \to \hat{x}^*(\pi)$ is a $*$-representation of A^* on H_π;*

(2) *the map $x^* \to \hat{x}^*$ is an $*$-isomorphism of the algebra A^* into the algebra $\mathscr{L}^\infty(\Sigma)$ such that $\|\hat{x}^*\| \le \|x^*\|$.*

Proof. (1) for any $\pi \in \Sigma$, suppose that H_π is the representation space of π. Choose an orthonormal basis $\{e_i^\pi\}$ for H_π. Assume that $\pi(e_i^\pi) = \sum_j a_{ij}^\pi \otimes e_j^\pi$. For any $x^*, y^* \in A^*$, we have

$$\begin{aligned}\langle (x^* \hat{*} y^*)(\pi)e_i^\pi, e_j^\pi\rangle &= (x^* * y^*)(\langle \bar{\pi}e_i^\pi, I \otimes e_j^\pi\rangle)\\ &= (x^* * y^*)(k(a_{ij}^\pi)) = (x^* \otimes y^*)\Phi(k(a_{ij}^\pi))\\ &= (y^* \otimes x^*)[(k \otimes k)\Phi(a_{ij}^\pi)] = \sum_k x^*(k(a_{kj}^\pi))y^*(k(a_{ik}^\pi)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \hat{x}^*(\pi)\hat{y}^*(\pi)e_i^\pi, e_j^\pi\rangle &= \langle \hat{y}^*(\pi)e_i^\pi, (\hat{x}^*(\pi))^* e_j^\pi\rangle\\ &= \sum_k y^*(k(a_{ik}^\pi))\langle e_k^\pi, (\hat{x}^*(\pi))^* e_j^\pi\rangle\\ &= \sum_k x^*(k(a_{kj}^\pi))y^*(k(a_{ik}^\pi)),\end{aligned}$$

so we have $(x^* \hat{*} y^*)(\pi) = \hat{x}^*(\pi)\hat{y}^*(\pi) \forall x^*, y^* \in A^*$.

Also, we have

$$\begin{aligned}\langle (\hat{x}^*(\pi))^* e_i^\pi, e_j^\pi\rangle &= \langle e_i^\pi, \hat{x}^*(\pi)e_j^\pi\rangle\\ &= \overline{\langle \hat{x}^*(\pi)e_j^\pi, e_i^\pi\rangle} = \overline{x^*(k(a_{ji}^\pi))} = x^*(a_{ij}),\end{aligned}$$

but

$$\begin{aligned}\langle \widehat{k^*(x^*)}(\pi)e_i^\pi, e_j^\pi\rangle &= k^*(x^*)(\langle \bar{\pi}e_i^\pi, I \otimes e_j^\pi\rangle)\\ &= k^*(x^*)(k(a_{ij}^\pi)) = x^*(a_{ij}^\pi).\end{aligned}$$

Thus we have $\widehat{k^*(x^*)}(\pi) = (\hat{x}^*(\pi))^*$. By the definition of $\hat{x}^*(\pi)$, we can easily see that $\|\hat{x}^*(\pi)\| \le \|x^*\|$.

In fact, $\forall \xi = \sum_i a_i e_i^\pi$, $\eta = \sum_j b_j e_j^\pi$, with $\Sigma_i |a_i|^2 \le 1$, $\sum_j |b_j|^2 \le 1$.

$$x^*(\langle \bar{\pi}\xi, I \otimes \eta\rangle) = x^*\left(\left\langle \sum_i a_i \bar{\pi}(e_i^\pi), I \otimes \sum_j b_j e_j^\pi \right\rangle\right)$$
$$= \sum_{i,j} a_i \overline{b_j} x^*(\langle \bar{\pi}(e_i^\pi), I \otimes e_j^\pi\rangle) = \sum_{i,j} a_i \overline{b_j} x^*(k(a_{ij}^\pi)).$$

Since π is a unitary representation, $\sum_j a_{ij}^\pi (a_{ij}^\pi)^* = I$. This implies that $||a_{ij}^\pi||^2 = ||a_{ij}^\pi (a_{ij}^\pi)^*|| \le ||I|| = 1$. Thus we have

$$|x^*(\langle \bar{\pi}\xi, I \otimes \eta\rangle)| \le \sum_{i,j} |a_i b_j| \, ||x^*|| \le ||x^*|| \left(\sum_i |a_i|^2\right)^{1/2} \left(\sum_j |b_j|^2\right)^{1/2} \le ||x^*||.$$

Hence $||\hat{x}^*(\pi)|| \le ||x^*||$.

(2) Since the operator norm of $\hat{x}^*(\pi)$ is the ϕ_∞ norm, we have

$$||\hat{x}^*||_\infty = \sup\{||\hat{x}^*(\pi)||_\infty : \pi \in \Sigma\} \le ||x^*||.$$

It remains to show that $x^* \to \hat{x}^*$ is one-to-one. Suppose that $x^* \ne 0$. Since A has the Peter-Weyl property, there are $\pi \in \Sigma$ and a_{ij}^π such that $x^*(a_{ij}^\pi) \ne 0$. Hence by the definition of $\hat{x}^*(\pi)$ there exist $\xi, \eta \in H_\pi$ such that $\langle \hat{x}^*(\pi)\xi, \eta\rangle \ne 0$, which shows that $\hat{x}^* \ne 0$. This completes the proof.

Note that from the proof of Theorem 2.3 we can easily see that the Fourier transform maps $L^1(A)$ into $\mathscr{L}^\infty(\Sigma)$, which does not increase the norm, since $L^1(A)$ is a subspace of A^*.

For $\pi \in \Sigma$, let $\mathscr{I}_\pi(A)$ be the linear space spanned by $\{a_{ij}^\pi : 1 \le i, j \le d_\pi\}$ and $\mathscr{I}(A)$ be the linear space spanned by $\{a_{ij}^\pi : 1 \le i, j \le d_\pi, \pi \in \Sigma\}$. For $x \in \mathscr{A}$, $a \in A$, we define $\langle a, x\rangle = \sigma(ax)$. In the next result, we will make this assumption. Then we have

Proposition 2.4. *Let $\pi \in \Sigma$. Then:*

(1) $\{\hat{f}(\pi) : f \in \mathscr{I}_\pi(A)\} = B(H_\pi)$;

(2) $\{\hat{f} : f \in \mathscr{I}(A)\} = C_{00}(\Sigma)$.

Proof. (1) For a_{ij}^π, we have $\langle \widehat{a_{ij}^\pi}(\pi) e_k, e_l\rangle = a_{ij}^\pi(k(a_{kl}^\pi)) = \sigma(a_{ij}^\pi k(a_{kl}^\pi)) = 1/d_\pi \delta_{il}\delta_{jk}$, so $d_\pi \widehat{a_{ij}^\pi}(\pi)$ is the operator on H_π which sends e_k into $\delta_{jk} e_i$. But these operators span the space $B(H_\pi)$.

(2) It follows easily from (1), the definition of $\mathscr{I}(A)$, and $C_{00}(\Sigma)$.

Theorem 2.5. *The map $f \to \hat{f}$ is an inner product-preserving linear map of the Hilbert space $L^2(A)$ onto the Hilbert space $\mathscr{L}^2(\Sigma)$. For $f \in L^2(A)$, we have*

$$f = \sum_{\pi \in \Sigma} d_\pi \sum_{j,k} \langle \hat{f}(\pi) e_k^\pi, e_j^\pi\rangle a_{jk}^\pi,$$

where the notation is as above, and the series in the above expression converges in the L^2 metric.

Proof. For any $f \in \mathscr{A}$ and $\pi \in \Sigma$, we have

$$||\hat{f}(\pi)||_{\phi_2}^2 = \operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*) = \sum_{j,k} |\langle \hat{f}(\pi) e_k^\pi, e_j^\pi\rangle|^2 = \sum_{j,k} |\sigma(fk(a_{ij}^\pi))|^2.$$

Since A has the Peter-Weyl property, $\{d_\pi^{1/2} a_{ij}^\pi : 1 \le i, \ j \le d_\pi; \ \pi \in \Sigma\}$ form an orthonormal basis for $L^2(A)$ and

$$||f||^2 = \sum_{\pi\in\Sigma}\sum_{j,k} |\langle f, d_\pi^{1/2} a_{ij}^\pi\rangle|^2 = \sum_{\pi}\sum_{j,k} d_\pi |\sigma(f(a_{ij}^\pi)^*)|^2.$$

Thus we have

$$||\hat{f}||_2^2 = \sum_{\pi\in\Sigma} d_\pi ||\hat{f}(\pi)||_{\phi_2}^2 = ||f||_2^2.$$

By using the polar identity, we obtain $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$ for all $f, g \in \mathscr{A}$. Thus $f \to \hat{f}$ can be extended to a linear isometry of $L^2(A)$ into $\mathscr{L}^2(\Sigma)$. By Proposition 2.4, this map carries $\mathscr{I}(A)$ onto the dense subspace $C_{00}(\Sigma)$ of $\mathscr{L}^2(\Sigma)$, so the image of $L^2(A)$ is the whole space $\mathscr{L}^2(\Sigma)$.

Since $\langle f, a_{ij}^\pi\rangle = \tau(f(a_{ij}^\pi)^*) = \tau(fk(a_{ji}^\pi)) = \langle \hat{f}(\pi)e_j^\pi, e_i^\pi\rangle$, we have

$$f = \sum_{\pi\in\Sigma}\sum_{i,j} d_\pi \langle f, a_{ij}^\pi\rangle a_{ij}^\pi = \sum_{\pi\in\Sigma} d_\pi \sum_{j,k} \langle \hat{f}(\pi)e_k^\pi, e_j^\pi\rangle a_{jk}^\pi.$$

This completes the proof.

3. Fourier series in $L^2(A)$

For each $\pi \in \Sigma$, let H_π be the representation space of π and (a_{ij}^π) the matrix coefficient of π with respect to a fixed orthonormal basis $\{e_1^\pi, \dots, e_n^\pi\}$ of H_π.

It is straightforward to verify that for any $n \times n$ matrices A, B we have

$$\operatorname{tr}(AB^*) = \sum_{i,j} \langle Ae_i, e_j\rangle \overline{\langle Be_i, e_j\rangle},$$

where $\{e_i\}$ is an orthonormal basis for C^n. By using Theorem 2.5, we get the following Parseval formula.

Corollary 3.1. *For $f \in L^2(A)$, we have*

$$||f||_2^2 = \sum_\pi d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*).$$

Proof. Since for the orthonormal basis $\{e_i^\pi\}$ of H_π, for which π has matrix coefficient (a_{ij}^π), we have

$$\begin{aligned}\langle \hat{f}(\pi)e_i^\pi, e_j^\pi\rangle &= \langle (\sigma\otimes I)[(f\otimes I)\bar{\pi}e_i^\pi], e_j^\pi\rangle \\ &= \left\langle \sum_k \sigma(fk(a_{ik}^\pi))e_k^\pi, e_j^\pi \right\rangle = \sigma(fk(a_{ij}^\pi)) = \langle f, a_{ji}^\pi\rangle,\end{aligned}$$

we have $\operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*) = \sum_{i,j} |\langle f, a_{ij}^\pi\rangle|^2$. Thus by Theorem 2.5, we get $||f||_2^2 = \sum_\pi d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*)$.

As a consequence of Corollary 3.1, we have the following polarized form of Parseval formula:

$$\langle f, g\rangle = \sum_{\pi\in\Sigma} d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{g}(\pi)^*)$$

and the following Bessel inequality.

Corollary 3.2. *For any subset $P \subset \Sigma$, we have*

$$\sum_{\pi \in P} d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*) \leq \|f\|_2^2.$$

Finally, in this section we prove the following result. We will use it in our study of positive definite functions of Hopf C^*-algebras.

Theorem 3.3. *For any $f, g \in L^2(A)$, $\phi \in L^1(A)$, we have*

$$\langle f * \phi, g \rangle = \sum_\pi d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{\phi}(\pi)\hat{g}(\pi)^*).$$

Proof. By Corollary 3.1 and Theorem 2.3, we have

$$\langle f * \phi, g \rangle = \sum_\pi d_\pi \operatorname{tr}(f \hat{*} \phi(\pi)\hat{g}(\pi)^*) = \sum_\pi d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{\phi}(\pi)\hat{g}(\pi)^*).$$

4. Positive definite functions on Hopf C^*-algebras

In this section, we study positive definite functions on Hopf C^*-algebras. Our main purpose here is to show that the positive definiteness can be characterized by its Fourier transform.

Definition 4.1. A function $\phi \in L^1(A)$ is said to be positive definite if

$$\langle f * \phi, f \rangle \geq 0 \quad \text{for all } f \in \mathscr{A}.$$

Proposition 4.2. *A function $\phi \in L^1(A)$ is positive definite iff $\hat{\phi}(\pi)$ is positive and selfadjoint for every $\pi \in \Sigma$.*

Proof. ($\Rightarrow$) Suppose that $\phi \in L^1(A)$ is positive definite. Then by Theorem 3.3, we know that for any $f \in L^2(A)$

$$0 \leq \langle f * \phi, f \rangle = \sum_\pi d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{\phi}(\pi)\hat{f}(\pi)^*).$$

By the arbitrariness of $f \in L^2(A)$ and Proposition 2.4, we know that $\hat{\phi}(\pi)$ is positive definite for each $\pi \in \Sigma$.

($\Leftarrow$) This follows from Theorem 3.3.

Now if $\phi \in L^1(A)$ is a positive definite function on A, for $f, g \in \mathscr{A}$ define an inner product on $\mathscr{A}$ as

$$\langle f, g \rangle_\phi = \langle f * \phi, g \rangle.$$

Let $\mathscr{N}_\phi = \{f \in \mathscr{A} : \langle f, f \rangle_\phi = 0\}$. Then it is a subspace of $\mathscr{A}$, and $\langle \cdot, \cdot \rangle_\phi$ induces a norm on $\mathscr{A}/\mathscr{N}_\phi$. Let H_ϕ be the completion of $\mathscr{A}/\mathscr{N}_\phi$ under this norm. Then we have

Theorem 4.3. *With the same notation as above, we have the map $\Phi: \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ defines a left unitary representation of $\mathscr{A}$ on H_ϕ.*

Proof. For $\pi_1, \pi_2 \in \Sigma$, $a_{ij}^{\pi_1}, a_{kl}^{\pi_2}$ as above, we have

$$\begin{aligned}\langle \Phi(a_{ij}^{\pi_1}), \Phi(a_{kl}^{\pi_2})\rangle_\phi &= \delta_{\pi_1,\pi_2} \sum_{m,n=1}^{d_{\pi_1}} a_{im}^{\pi_1}(a_{kn}^{\pi_1})^* \langle a_{mj}^{\pi_1}, a_{nl}^{\pi_1}\rangle_\phi \\ &= \delta_{\pi_1,\pi_2} \sum_{m,n=1}^{d_{\pi_1}} a_{im}^{\pi_1}(a_{kn}^{\pi_1})^* \langle a_{mj}^{\pi_1} * \phi, a_{nl}^{\pi_1}\rangle \\ &= \delta_{i,k}\delta_{\pi_1,\pi_2} \sum_{m,n=1}^{d_{\pi_1}} (\sigma\otimes\sigma)(a_{mj}^{\pi_1}\otimes\phi, \Phi(a_{nl}^{\pi_1})^*) \\ &= 1/d_{\pi_1}\delta_{i,k}\delta_{\pi_1,\pi_2}\sigma(\phi(a_{jl}^{\pi_1})^*).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle a_{ij}^{\pi_1}, a_{kl}^{\pi_2}\rangle_\phi &= \langle a_{ij}^{\pi_1} * \phi, a_{kl}^{\pi_2}\rangle \\ &= (\sigma\otimes\sigma)[(a_{ij}^{\pi_1}\otimes\phi)\Phi(a_{kl}^{\pi_2})^*] \\ &= \sum_{m=1}^{d_{\pi_2}} \sigma(a_{ij}^{\pi_1}(a_{km}^{\pi_2})^*)\sigma(\phi(a_{ml}^{\pi_2})^*) \\ &= 1/d_{\pi_1}\delta_{\pi_1,\pi_2}\delta_{i,k}\sigma(\phi a_{jl}^{\pi_1}).\end{aligned}$$

Here we repeatedly use the orthogonal relations of matrix coefficients of finite-dimensional irreducible representations. Thus the map $f \to \Phi(f)$ can be extended to a unitary representation of $\mathscr{A}$ on H_ϕ.

5. Smooth positive definite functions of Hopf C^*-algebras

In this section we study smooth positive definite functions on Hopf C^*-algebras. First of all, we introduce an equivalence relation on smooth positive definite functions. Then we study the representation of smooth positive definite functions under the equivalence relation and determine the extreme points of the set of all normalized smooth positive definite functions. A Bochner-type theorem for smooth positive definite functions is also proved. But unlike the result for the compact group case, we show that under the equivalence relation the representation is unique.

Definition 5.1. A positive definite function on the Hopf C^*-algebra (A, Φ, k, e) is called smooth if it belongs to $\mathscr{A}$.

Let f_1, f_2 be smooth positive definite functions on A. We say that f_1, f_2 are equivalent if

$$\langle f * f_1, f\rangle = \langle f * f_2, f\rangle \quad \forall f \in \mathscr{A}.$$

We denote it by $f_1 \doteq f_2$. Obviously it induces an equivalence relation on smooth positive definite functions. We are going to study the representation of smooth positive definite functions under this equivalence relation. We start with the following simple lemma.

Lemma 5.2. *For any $\pi \in \Sigma$, we have $a_{ij}^{\pi} \doteq 0$ iff $i \neq j$.*

Proof. First we show that if $i \neq j$, $a_{ij}^{\pi} \doteq 0$. In fact, for any $\pi_1 \in \Sigma$, $a_{kl}^{\pi_1}$, we have

$$\begin{aligned}\langle a_{kl}^{\pi_1}, a_{kl}^{\pi_1}\rangle_{a_{ij}^{\pi}} &= (\sigma \otimes \sigma)[(a_{kl}^{\pi_1} \otimes a_{ij}^{\pi})\Phi(a_{kl}^{\pi_1})^*] \\ &= \sum_{m=1}^{d_{\pi_1}} \sigma(a_{kl}^{\pi_1}(a_{km}^{\pi_1})^*)\sigma(a_{ij}^{\pi}(a_{ml}^{\pi_1})^*) \\ &= 1/d_{\pi_1}\sigma(\sigma_{ij}^{\pi}(a_{ll}^{\pi_1})^*) = (1/d_{\pi_1})^2\delta_{\pi,\pi_1}\delta_{i,l}\delta_{j,l} = 0,\end{aligned}$$

since $i \neq j$. Thus we have $\langle f, f\rangle_{a_{ij}^{\pi}} = 0$ for all $f \in \mathscr{A}$.

On the other hand, by the above calculation, we can easily see that $\langle a_{ii}^{\pi}, a_{ii}^{\pi}\rangle_{a_{ii}^{\pi}} \neq 0$; thus $a_{ii}^{\pi} \not\doteq 0$. This completes the proof.

Now we are ready to prove one of the main results of this section.

Theorem 5.3. *Let f be a smooth positive definite function on A. Then we have that*

$$f \doteq \sum_{\pi}\sum_{i=1}^{d_{\pi}} c_i^{\pi} a_{ii}^{\pi},$$

where $c_i^{\pi} \geq 0\ \forall i$ and only finitely many c_i^{π} are nonzero.

Proof. Since $f \in \mathscr{A}$, it is a finite linear combination of the elements of matrix coefficients. By Lemma 5.2, we know that the terms from the off diagonal of the matrix coefficients are zero under the equivalence relations, so f can be written as above. Since f is positive definite, by the orthogonal relations of matrix coefficients, we have that the coefficients in the representation have to be nonnegative. This finishes the proof.

As a consequence of Lemma 5.2 and Theorem 5.3, we have

Corollary 5.4. *For every smooth positive definite function f on A, there exist a finite dimensional representation π of A on a Hilbert space H_{π} and an element $x \in H_{\pi}$ such that*

$$f \doteq \langle \pi(x), I \otimes x\rangle.$$

The converse is also true.

Proof. Suppose that f is a smooth positive definite function. Then by Theorem 5.3, we have

$$f \doteq \sum_{\pi_l}\sum_{i=1}^{d_{\pi_l}} c_i^{\pi_l} a_{ii}^{\pi_l},$$

where $c_i^{\pi_l} \geq 0\ \forall i$ and only finite many c_i^{π} are nonzero. Let π be the direct sum of the all π_l and H_{π} the subspace of $L^2(A)$ spanned by the elements $\{a_{ij}^{\pi_l}\}$, where π_l are the elements of Σ appearing in the above decomposition of f with nonzero coefficients. Then obviously π is a finite-dimensional representation of A on H_{π}. Let $x = \sum_{\pi_l}\sum_i (c_i^{\pi_i})^{1/2} a_{ii}^{\pi_l}$. Then by the orthogonal relation of the matrix coefficients of the finite-dimensional irreducible representations, we have

$$f \doteq \langle \pi(x), I \otimes x\rangle.$$

Conversely, suppose that $f \doteq \langle \pi(x), I \otimes x\rangle$ for some finite-dimensional representation π of A, so π can be written as the direct sum of finitely many

elements of Σ. Then by Lemma 5.2, we have $f \doteq \sum_\pi \sum_{i=1}^{d_\pi} c_i^\pi a_{ii}^\pi$, where $c_i^\pi \geq 0$ $\forall i$ and only finite many c_i^π are nonzero. Thus we finish the proof.

Next we study the elementary smooth positive definite functions on Hopf C^*-algebras. Let $\mathscr{P}(A)$ denote the set of all smooth positive definite functions on A. Obviously we have $\mathscr{P}(A) + \mathscr{P}(A) \subset \mathscr{P}(A)$. We can introduce a partial ordering $\ll$ on $\mathscr{P}(A)$ as $f \ll g$ iff $f, g \in \mathscr{P}(A)$ and $g - f \in \mathscr{P}(A)$. Now a natural question arises: which elements of $\mathscr{P}(A)$ are minimal with respect to this partial ordering, i.e., all those elements $f \in \mathscr{P}(A)$, which are nonzero and for which $g \in \mathscr{P}(A)$, $g \ll f \Rightarrow g = cf$, for some number $c \geq 0$. We call the minimal smooth positive definite functions elementary smooth positive definite functions. Then we have the following simple characterization for elementary smooth positive definite functions.

Theorem 5.5. *A nonzero element f in $\mathscr{P}(A)$ is an elementary smooth positive definite function iff $f \doteq ca_{ii}^\pi$ for some $\pi \in \Sigma$, $i \in \{1, \dots, d_\pi\}$, and nonnegative constant c.*

Proof. If $f \in \mathscr{P}(A)$ is an elementary smooth positive definite function, then by Theorem 5.3 we have that $f \doteq \sum_\pi \sum_{i=1}^{d_\pi} c_i^\pi a_{ii}^\pi$, where $c_i^\pi \geq 0$ $\forall i$ and only finitely many c_i^π are nonzero. Since f is elementary, it has at most one of the c_i's nonzero.

Conversely, if $f \doteq ca_{ii}^\pi$ for some $\pi \in \Sigma$, $i \in \{1, \dots, d_\pi\}$, and nonnegative constant c, then, by the orthogonal relations of matrix coefficients of finite-dimensional irreducible representations and Theorem 5.3, we can easily verify that f is an elementary smooth positive definite function. This completes the proof.

Finally we turn our attention to the Bochner-type result for Hopf C^*-algebras. We will show that under the equivalence relation defined as above the representation of smooth positive definite functions is unique. Before doing this, let us recall a definition. An element $f \in \mathscr{P}(A)$ is said to be normalized if $e(f) = 1$. Then we have

Proposition 5.6. *Let $f \in \mathscr{P}(A)$ be a normalized smooth positive definite function. Then we have $f \doteq \sum_\pi \sum_{i=1}^{d_\pi} c_i^\pi a_{ii}^\pi$, where $c_i^\pi \geq 0$ $\forall i$, only finitely many c_i^π are nonzero, and $\sum_\pi \sum_i c_i^\pi = 1$.*

Proof. Since $e(a_{ij}^\pi) = \delta_{ij}$ for any $\pi \in \Sigma$, the conclusion follows easily from Theorem 5.3.

Let $\mathscr{N}(A)$ denote the set of all normalized smooth positive definite functions on A. Then obviously we can see that it is a convex set. Then under the equivalence relation, we have the following.

Proposition 5.7. *The extreme points of $\mathscr{N}(A)$ consist of the set $\{a_{ii}^\pi : 1 \leq i \leq d_\pi, \pi \in \Sigma\}$.*

Proof. It follows easily from Theorem 5.5 and Proposition 5.6.

Note that we can see that the set of extreme points of $\mathscr{N}(A)$ is exactly the set of normalized elementary smooth positive definite functions on A. Thus we get the following Bochner-type result for smooth positive definite functions.

Theorem 5.8. *For any $f \in \mathscr{P}(A)$, we have $f \doteq \sum_i c_i \phi_i$, where the ϕ_i are normalized elementary smooth positive definite functions on A and c_i are nonnegative numbers, finitely many not equal to zero. Such a decomposition is unique.*

Proof. The first part follows from Theorem 5.3 and the previous remark. The uniqueness follows from Lemma 5.2 and the orthogonal relations of matrix coefficients of finite-dimensional irreducible representations.

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