PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 2, February 1995

AN EXTENSION OF THE FORAN INTEGRAL

SHUSHENG FU

(Communicated by Andrew Bruckner)

ABSTRACT. In this note, we shall introduce a new integral, AF integral, which includes the Foran integral and the Kubota integral.

The general Denjoy integral can be extended in at least two ways, namely, to the Kubota integral [2] and the Foran integral [1]. It is an easy exercise to check that the two integrals do not include each other. The purpose of this short note is to extend the Foran integral to the approximately continuous Foran integral. We call it the AF integral, which includes the Foran integral and the Kubota integral.

Throughout the sequel we will consider the real valuable functions which are defined on the closed interval $I = [a, b]$.

Definition. Given a natural number N and a subset E of an interval I, a function F will be said to be $A(n)$ on E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $I_1, \dots, I_k, \dots$ are nonoverlapping intervals with $E \cap I_k \neq \varnothing$ and $\sum_k |I_k| < \delta$, there exist intervals J_{kn}, $n = 1, 2, \dots, N$, such that

$$B\left(f; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^{N} I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon.$$

The class AF will consist of all approximately continuous functions for which there exists a sequence of sets E_n and natural numbers N_n such that $I = \bigcup_n E_n$ and F is $A(N_n)$ on E_n. The class D will consist of all approximately continuous functions which are approximately differentiable at almost all points. We list the statement of a known theorem which will be used in the paper.

O'Malley's monotonicity theorem asserts: If (i) G is in Baire Class 1 on I, (ii) $\overline{\lim}_{x\to x_0^-} \operatorname{ap} G(x) \le G(x_0) \le \overline{\lim}_{x\to x_0^+} \operatorname{ap} G(x)$ for each x_0 in I, and (iii) the interior of $G\{x : G^+_{\mathrm{ap}}(x) \le 0\}$ is empty, then G is nondecreasing. In particular, approximately continuous functions satisfy (i) and (ii).

Lemma. *Let F be a function of the class $AF \cap D$, and suppose $F'_{\mathrm{ap}}(x) \ge 0$ a.e. Then F is nondecreasing.*

Received by the editors January 6, 1993 and, in revised form, April 28, 1993.
1991 *Mathematics Subject Classification.* Primary 26A39; Secondary 26A24.
Key words and phrases. Foran integral, Kubota integral, O'Malley monotone theorem.
The author was partially supported by a Fujian Science Foundation.

Proof. Let $G(x) = F(x) + \varepsilon x$, where $\varepsilon > 0$; then $G \in AF \cap D$ can be found in [1, property (iv)]. From Foran's result [1, property (iii)] it follows that G satisfies Lusin's condition (N) on I. This implies that $mG\{x : G^+_{\text{ap}}(x) \leq 0\} = 0$. Now it is clear that G satisfies three conditions of the O'Malley monotone theorem [3, Theorem 1]. Therefore, $G(x)$ is a nondecreasing function on I. Let $\varepsilon \to 0$; this yields that F is nondecreasing. □

It is routine to show the following theorem.

Theorem. *Let F be a function of the class $AF \cap D$, and suppose $F'_{\text{ap}}(x) = 0$ a.e. Then F is a constant.*

Clearly, the class $AF \cap D$ is an additive class, so $AF \cap D$ can be taken as a class of primitives and the AF integral can be defined by

$$AF \int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'_{\text{ap}}(x) = f(x)$ a.e. on I. The uniqueness of the integration is implied by the above theorem.

Then if $F'_{\text{ap}}(x) = G'_{\text{ap}}(x)$ a.e. on I, it follows that $(F - G)'_{\text{ap}} = 0$ a.e. on I and thus $F - G$ is a constant. Consequently

$$AF \int_a^b f(x)\,dx = F(b) - F(a) = G(b) - G(a).$$

A function F is a primitive for the Kubota integral on I if F is AC on each set in a sequence of the sets whose union is I, F'_{ap} exists a.e. and F is approximately continuous on I. From the relations between AC and $A(N)$ on a set [1, property (i)] and the definition of the Kubota integral it is now clear that the AF integral includes the Kubota integral. Evidently the AF integral includes the Foran integral.

Acknowledgment

The author thanks the referee for his suggestions. The author also expresses his sincere gratitude to Professor Peng Yee Lee for discussing with him the problem of this work.

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Department of Mathematics, University of Fuzhou, Fuzhou, Fujian, People's Republic of China