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SPLINE WAVELET BASES OF WEIGHTED L^p SPACES, $1 \le p < \infty$

J. GARCÍA-CUERVA AND K. S. KAZARIAN

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ABSTRACT. We study necessary conditions on the weight w for the spline wavelet systems to be bases in the weighted space $L^p(w)$.

In this article we study some questions arising in the investigation of the problem of describing those classes of weight functions w for which the wavelet systems are bases or unconditional bases in the weighted spaces $L^p(w) = L^p(R, w)$. A priori these conditions depend on the concrete system. But there are some features which are common to all wavelet systems. Here we examine only the case of spline wavelet systems defined on the real line R.

For a given non-negative integer m, the spline wavelets are defined in the following way:

$$\text{Let}\quad V_0 = \{f \in L^2(R) \cap C^{m-1}(R) \quad \text{such that the restriction of } f \text{ to each interval }]n, n+1[\text{ is a polynomial of degree} \le m\},$$

where by $C^r(R)$ we denote the class of functions on R whose derivatives of order r are continuous and by $C^{-1}(R)$ we denote the class of piecewise continuous functions on R. Defining $V_{j+1} = \{f(2x) : f(x) \in V_j\}$, we get a multiscale analysis of $L^2(R)$ in the sense of Mallat and Meyer (see [M], [D]). Let W_j be the orthogonal complement of V_j in V_{j+1}. It is well known (see [S], [M], [D]) that there is a function $\psi \in V_1$ such that the functions $\psi(x-k)$, $k \in Z$, form an orthonormal basis of W_0, and consequently the system

$$\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in Z\}$$

is an orthonormal basis of $L^2(R)$. ψ can even be chosen so that it satisfies the following regularity property:

$$|\psi^{(l)}(x)| \le C_{M,l}(1+|x|)^{-M} \tag{1}$$

for all $M > 0$ and $0 \le l \le m$.

Here we are going to study only necessary conditions on the weight w so that the system $\{\psi_{j,k}\}_{j,k\in Z}$, for some enumeration of the indices, is a basis

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in the weighted space $L^p(w)$, $1 \le p < \infty$. Sufficiency results for the system $\{\psi_{j,k}\}_{j,k\in Z}$ to be an unconditional basis in $L^p(w)$, $1 < p < \infty$ (namely that $w \in A_p$, class of weights to be defined below) follow from the Calderón-Zygmund theory for regular singular integrals. See the paper [G-K] where the authors develop such theory (including the corresponding results for weighted Hardy spaces $H^p(w)$, $0 < p \le 1$) from simple atomic estimates.

The main part of the proof of the basisness of some classical systems in the weighted spaces is the proof of the uniform boundedness of the partial sums in the weighted norm, which in its turn can be derived from the boundedness in the weighted spaces of such operators as the conjugate function, the Hilbert transform, the Hardy-Littlewood maximal function, the square function, etc. Of course, the work of Calderón and Zygmund was basic to understand these operators, but the history of the weighted estimates can be traced to the works of Hardy and Littlewood, Babenko, Stein, Hirschman, Gaposkin, Helson and Szegö, and many others. The combination of the Calderón-Zygmund theory with the fundamental work of B. Muckenhoupt [Mu] led to the development of a complete machinery to get weighted results.

The classes of weights discovered by Muckenhoupt are usually denoted by A_p. They play a basic role in Fourier Analysis, and they also arise naturally in the problem we are investigating. The main properties of these classes of weights are studied systematically in [G-R]

Studying the problem of basisness in the weighted spaces $L^p(w)$, $1 \le p < \infty$, one comes across a new phenomenon which does not arise for the classical complete orthonormal systems defined on finite intervals (see [Z]). In this article we are going to concentrate our attention mainly on this new obstacle. The following theorem is one of the main tools for clarifying matters.

Theorem 1. *Let Φ be a locally integrable function on R such that:*

$$\int_R \frac{|\Phi(x)|}{(1+|x|)^N}\,dx < \infty \qquad \text{for some } N > 0 \tag{2}$$

and

$$\int_R \Phi(t)\psi_{j,k}(t)\,dt = 0 \qquad \text{for every} \quad j, k \in Z, \tag{3}$$

where $\psi_{j,k}$ are the spline wavelets of order m considered above. Then on each half-line $R_- =]-\infty, 0]$ and $R_+ = [0, +\infty[$, the function Φ is almost everywhere equal to some polynomial of degree $\le m$.

Proof. Let n be any nonnegative integer. Consider the segment $[0, 2^n]$, and let f be an arbitrary continuous function defined on $[0, 2^n]$. Denote by $P^{(n)}$ the projection operator $P^{(n)} : L^2[0, 2^n] \to T_n$, where T_n is the subspace of $L^2[0, 2^n]$ consisting of the polynomials of degree $\le m$ restricted to the segment $[0, 2^n]$. From the fact that $\psi \in V_0$, we immediately obtain:

$$\int_0^{2^n} [f(t) - P^{(n)}(f)(t)]\psi_{j,k}(t)\,dt = 0 \qquad \text{for every} \quad j < -n, k \in Z. \tag{4}$$

Hence, by the orthonormality of $\{\psi_{j,k}\}_{j,k\in Z}$ we have:

$$\sum_{j=-n}^{\infty} \sum_{k=-\infty}^{+\infty} a_{j,k}\psi_{j,k}(x) = f(x) - P^{(n)}(f)(x) \tag{5}$$

where $a_{j,k} = \int_0^{2^n} [f(t) - P^{(n)}(f)(t)]\psi_{j,k}(t)\,dt$ and the series in the left-hand side of (5) converges in $L^2(R)$. Actually this series converges in a stronger sense as we can see with the help of the following lemma, which is a slight modification of a result of Z. Cisielski and J. Domsta [C-D].

Lemma CD. *Let* P_ν *be the projection operator* $P_\nu : L^2(R) \to V_\nu$. *Then for every* $f \in C[0, 2^n]$, $\|P_\nu(f)\|_\infty \le C_m \|f\|_\infty$ *where* $C_m > 0$ *depends only on* m; $P_\nu(f)(x) \to f(x)$ *uniformly on* $[0, 2^n]$ *as* $\nu \to +\infty$; *and* $P_\nu(f)(x) \to 0$ *uniformly on* $]-\infty, -\delta[\cup]2^n + \delta, +\infty[$ *for every* $\delta > 0$ *as* $\nu \to +\infty$.

This lemma follows immediately from the results on P.7 of [C-D] (pp. 216–217) if we observe that the functions $g(2^\nu x - k)$, $k = -m+1, \dots, 2^{n+\nu} - 1$, where $g = \chi * \chi * \cdots * \chi$ ($m+1$ times) and χ is the characteristic function of $[0, 1]$, form an algebraic basis of the space $V_\nu^{(n)}$ consisting of the restrictions of the functions $\phi \in V_\nu$ to the segment $[0, 2^n]$ (see [Ch, pp. 81–87]).

In order to finish the proof of Theorem 1, we also need the following lemma:

Lemma 1. *Let*

$$K(x, y) = \sum_{k \in Z} \psi(x-k)\psi(t-k). \tag{6}$$

Then the following estimates hold:

$$\int_R |K(x, t)|\,dt \le C, \qquad x \in R; \tag{7}$$

$$|K(x, t)| \le C_M (1 + |x-t|)^{-M}, \tag{8}$$

where $M \ge 1$ *is arbitrary and the positive numbers* C *and* C_M *are independent of* x *and* t.

Proof of Lemma 1. The inequality (7) follows immediately from (6) and (1). In order to prove (8), assume that $x - t \ge 1$.

$$|K(x, t)| \le C \sum_{k=-\infty}^{+\infty} (1 + |x-k|)^{-(M+1)} (1 + |t-k|)^{-(M+1)}.$$

Hence, summing first for $k \ge x$ and writing $|x-t|$ instead of $|t-k|$, we obtain:

$$\sum_{k \ge x} (1 + |x-k|)^{-(M+1)} (1 + |t-k|)^{-(M+1)} \le C(1 + |x-t|)^{-(M+1)}. \tag{9}$$

The same estimate holds for the sum in $k \le t$. To estimate the remaining sum, note that the number of points k belonging to the interval $]t, x[$ is not larger than $1 + |x-t|$ and that the distance of every such point from one of the endpoints of the interval is $\ge |x-t|/2$. Combining these observations with (9), we get:

$$\begin{aligned} |K(x, t)| &\le C[(1 + |x-t|)^{-(M+1)} + (1 + (1/2)|x-t|)^{-(M+1)}(1 + |x-t|)] \\ &\le C(1 + |x-t|)^{-M}, \end{aligned}$$

and this completes the proof of Lemma 1.

Now we proceed with the proof of Theorem 1.

By Lemma 1 we see that the series $\sum_{k=-\infty}^{+\infty} a_{\nu,k}\psi_{\nu,k}(x)$ converges uniformly to $P_\nu(g) - P_{\nu-1}(g)$, where $g = f - P^{(n)}(f)$, and applying inequality (8) with $M = N + 2$ by Lebesgue-dominated convergence theorem, conditions (3), and Lemma CD we obtain:

$$0 = \int_0^{2^n} \Phi(t)[f(t) - P^{(n)}(f)(t)]\,dt = \int_0^{2^n} [\Phi(t) - P^{(n)}(\Phi)(t)]f(t)\,dt.$$

Hence $\Phi(t) = P^{(n)}(\Phi)(t)$ for a.e. $t \in [0, 2^n]$.

Since the integer $n \geq 0$ is arbitrary, we obtain that $\Phi(t) = P(t)$ a.e. on R_+, where P is a polynomial of degree $\leq m$.

We could proceed in the same way in the half-line R_-. □

Let

$$(x)_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We can easily derive from (1) that the functions $1, x, \ldots, x^{m-1}, (x)_+^m$, $(-x)_+^m$ are orthogonal to the functions $\psi_{j,k}$ $(j, k \in Z)$. To see that, take one of the mentioned functions, denote it by ϕ, and integrate it against any of the functions $\psi(x-k)$ $(k \in Z)$. That this integral is 0 can be seen by truncating the function ϕ outside a large compact set and extending it to R in such a way that the resulting function belongs to V_0. For this new function, the integral against $\psi(x-k)$ is, of course, 0. By (1) the difference of the two integrals will be arbitrarily small if the compact set is chosen large enough. Finally we see that ϕ is orthogonal to all the functions $\psi_{j,k}$ $(j, k \in Z)$ by using the following property of the function ϕ : for every $a > 0$ $\phi(ax) = b_a\phi(x)$, where $b_a > 0$ depends only on $a > 0$ and ϕ.

Denote by U_m the set of those locally integrable functions which satisfy the conditions (2) and (3) for a given m. By Theorem 1 one can easily conclude that U_m is a linear space of dimension $\leq 2m + 2$ (actually we can prove that the dimension is $m + 2$, but in this article we do not use that fact).

We omit the proof of the following lemma because it is entirely similar to that of the corresponding result in [K, pp. 38–40].

Lemma 2. *Let $w \geq 0$ be a locally integrable function satisfying the same condition (2) as Φ in Theorem 1, and let $1 \leq p < \infty$.*

For the system $\{\psi_{j,k}\}_{j,k\in Z}$ of m-spline wavelets to be complete and/or minimal in the space $L^p(w)$ it is necessary and sufficient that the following conditions (i) and/or (ii) respectively are fulfilled:

(i) *A function of the form $w^{-1}u$, where $u \in U_m$, belongs to $L^{p'}(w)$ $(p^{-1} + p'^{-1} = 1$ and $p' = \infty$ for $p = 1)$ if and only if $u = 0$ a.e.*

(ii) *For every $j, k \in Z$ there exists a uniquely determined function $u_{j,k} \in U_m$, such that $\xi_{j,k} = (\psi_{j,k} + u_{j,k})w^{-1} \in L^{p'}(w)$.*

Definition 1. We will say that the weight function $w \geq 0$ has a singularity of order p at $+\infty$ (resp., $-\infty$) if for every $a > 0$ (resp., $b < 0$)

$$\int_a^{+\infty} w^{-\frac{1}{p-1}}\,dt = +\infty \qquad \left(\textit{resp.}, \int_{-\infty}^{b} w^{-\frac{1}{p-1}}\,dt = +\infty\right).$$

Theorem 2. *Let $1 \leq p < \infty$, and let $w \geq 0$ be a nonnegative function which satisfies the same condition (2) as Φ in Theorem 1 and also has singularities of* *order p at both $+\infty$ and $-\infty$. Assume also that ψ is the m-spline wavelet considered above with $m \geq 1$, so that $\psi \in C(R)$. Then if the system $\{\psi_{j,k}\}_{j,k\in Z}$ for some enumeration of the indices constitutes a Schauder basis in the space $L^p(w)$, we necessarily have that $w \in A_p$.*

Proof. By Theorem 1 and our assumptions on the function w it is obvious that

$$fw^{-1} \in L^{p'}(w) \quad \text{for } f \in U_m \qquad \text{iff} \qquad f = 0 \quad \text{a.e.}$$

Hence by Lemma 2 we obtain that the conjugate system of the system $\{\psi_{j,k}\}_{j,k\in Z}$ is $\{\psi_{j,k}w^{-1}\}_{j,k\in Z}$. The coefficients of the expansion of the function $f \in L^p(w)$ with respect to the system $\{\psi_{j,k}\}_{j,k\in Z}$ are defined by the equations

$$a_{j,k}(f) = \int_R f(t)\psi_{j,k}(t)\,dt, \qquad j, k \in Z. \tag{10}$$

By the Banach theorem we have that the partial sum operators are uniformly bounded; hence, we conclude that there is a number $C > 0$ independent of f such that

$$|a_{j,k}(f)|\,\|\psi_{j,k}\|_{L^p(w)} \leq C\|f\|_{L^p(w)} \quad \text{for every } j, k \in Z. \tag{11}$$

From (10) and (11), using the fact that $\|\psi\|_{L^{p'}} = \sup|\int_R f\psi\,dt|$, where the supremum is taken over the ball $\|f\|_{L^p} \leq 1$, we easily obtain that

$$\left(\int_R |\psi_{j,k}|^{p'} w^{1-p'}\,dt\right)^{1/p'} \|\psi_{j,k}\|_{L^p(w)} \leq C \quad \text{if } p > 1 \tag{12}$$

and

$$\|\psi_{j,k}w^{-1}\|_{L^\infty}\|\psi_{j,k}\|_{L^1(w)} \leq C \quad \text{if } p = 1. \tag{13}$$

The following observation will be useful in the sequel.

Claim. *For every $x \in R$ there is an integer $k \in Z$ such that $\psi(x-k) \neq 0$.*

From the contrary assumption we obtain that there exists some point $x_0 \in [0, 1[$ such that, for every function $f \in W_0$ and every $k \in Z$, $f(x_0 - k) = 0$. It is obvious that the point $y_0 = 1 - x_0$ will have the same property, that is, $f(y_0 - k) = 0$ for every $k \in Z$ and every $f \in W_0$. But there are known constructions of functions which belong to W_0 and have no such property, for example, the compactly supported splines constructed in [Ch-W]. Hence, our claim is true.

From the definition of the function ψ, it is obvious that there is an interval $]n, n+1[$, $n \in Z$, such that on each half of it, ψ is a nontrivial polynomial. Without loss of generality we can assume that ψ has only finitely many zeros $\{x_i\}_{i=1}^s$ $(0 \leq s \leq 2m)$ on the segment $[0, 1]$. According to our claim, for every i $(1 \leq i \leq s \leq 2m)$, one can find an integer k_i such that $\psi(x_i - k_i) \neq 0$. Hence, there is a positive number $\alpha > 0$ such that we can cover the segment $[0, 1]$ with open sets Δ_i, $0 \leq i \leq s$, where for $1 \leq i \leq s$, Δ_i is an open interval centered around x_i on which $|\psi(x - k_i)| > \alpha$ and $\Delta_0 = \{x \in]-1, 2[: |\psi(x)| > \alpha\}$, obviously a finite union of open intervals. Consequently there exists some $\epsilon > 0$ such that every open interval of length less than ϵ which contains some point of $[0, 1]$ is entirely contained in one of the sets $\{\Delta_i\}_{i=0}^s$. Hence, by (12) or

(13), using only translations for $j = 0$, one can deduce that there is a constant $C > 0$ such that for every interval $I \subset R$, whose length satisfies $\frac{\epsilon}{2} \leq |I| < \epsilon$, we have

$$\frac{1}{|I|} \int_I w \, dt \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} dt \right)^{p-1} \leq C \quad \text{if} \quad p > 1 \tag{14}$$

or

$$\frac{1}{|I|} \int_I w \, dt \, ||w^{-1}||_{L^\infty(I)} \leq C \quad \text{if} \quad p = 1. \tag{15}$$

For an arbitrary interval $I \subset R$ we find an integer $j \in Z$ such that $2^{j-1}\epsilon \leq |I| < 2^j\epsilon$. Then dilating by 2^j and translating the sets $\{\Delta_i\}_{i=0}^r$ we can cover the interval I by one of the resulting sets. Hence, using conditions (12) or (13) for j and respectively k, we immediately get conditions (14) or (15) with the same $C > 0$ on the right-hand side. □

If in Theorem 2 we do not assume that the weight function w has singularities of order p at $\pm\infty$, then we get more complicated necessary conditions which probably are also sufficient. These questions will be discussed in a forthcoming publication. Here we state the necessary and sufficient conditions for the Haar wavelet system to be an unconditional basis in the weighted space $L^p(w)$, $1 < p < \infty$. The proof of the following theorem can be obtained in the same way as the proof of the corresponding result for the Haar system in the classical setting of the segment $[0, 1]$, which is given in [K2]. Denote

$$\psi^{(0)}(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *Let $1 < p < \infty$, and let $w \geq 0$ be a weight function on R. Consider the system $\Psi = \{\psi_{j,k}^{(0)}(x) = \psi^{(0)}(2^j x - k)\}_{j,k \in Z}$. Then the following conditions are equivalent for w:*

(a) *The system Ψ, for some enumeration of the indices, is a Schauder basis of the space $L^p(w)$.*

(b) *Ψ is an unconditional basis of the space $L^p(w)$.*

(c) *On each of the halflines R_+ and R_-, the function w satisfies this property: Either w belongs to the class dyadic A_p on the halfline or else there is a sequence of dyadic intervals $\{\Delta_i\}_{i=-\infty}^{+\infty}$ contained in the halfline such that $|\Delta_i| = 2^{-i}$, $\Delta_i \subset \Delta_{i-1}$, and*

$$\frac{1}{|\Delta_i|} \int_{\Delta_i} w(t) \, dt \left(\frac{1}{|\Delta_i|} \int_{\Delta_i^c} w(t)^{-\frac{1}{p-1}} dt \right)^{p-1} \leq C \tag{16}$$

(where Δ_i^c is the complement of Δ_i with respect to the corresponding halfline), and for every dyadic interval not belonging to the collection $\{\Delta_i\}_{i=-\infty}^{+\infty}$ condition (14) holds with the same constant $C > 0$ appearing in (16), which is, of course, independent of the particular interval.

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DEPARTAMENTO DE MATEMÁTICAS, C-XV, UNIVERSIDAD AUTÓNOMA, 28049-MADRID, SPAIN
E-mail address, J. García-Cuerva: `cuerva@ccuam3.sdi.uam.es`
E-mail address, K. S. Kazarian: `kazarian@ccuam3.sdi.uam.es`