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ON THE APPROXIMATION OF FIXED POINTS FOR LOCALLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. Let X and its dual X^* be uniformly convex Banach spaces, D an open and bounded subset of X, T a continuous and pseudo-contractive mapping defined on $\mathrm{cl}(D)$ and taking values in X. If T satisfies the following condition: there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all x on the boundary of D, then the trajectory $t \to z_t \in D$, $t \in [0, 1)$, defined by $z_t = tT(z_t) + (1-t)z$ is continuous and converges strongly to a fixed point of T as $t \to 1^-$.

1. INTRODUCTION

Let X be a real Banach space, and let D be a subset of X. An operator $T : D \to X$ is said to be *k-pseudo-contractive* $(k > 0)$ (see [9]) if for each x, $y \in D$ and $\lambda > k$

$$(1) \qquad (\lambda - k)\|x - y\| \leq \|\lambda(x - y) - (Tx - Ty)\|.$$

For $k = 1$ $(k < 1)$ such mappings are said to be *pseudo-contractive* (respectively, *strongly pseudo-contractive*). By letting $r = 1/(\lambda - 1)$ and $k = 1$ in (1), we may derive the original definition of pseudo-contractive mappings, due to Browder [1], as follows:

$$(2) \qquad \|x - y\| \leq \|(1 + r)(x - y) - r(Tx - Ty)\|$$

holds for all x, $y \in D$ and all $r > 0$. However, by taking a semi-inner product approach (see also Kato [6]) we may describe (2) by

$$\langle Tx - Ty, j \rangle \leq \|x - y\|^2$$

for some $j \in J(x - y)$. The mapping $J : X \to 2^{X^*}$ is called the normalized duality mapping which is defined by

$$J(x) = \{j \in X^* : \langle x, j \rangle = \|x\|^2, \|j\| = \|x\|\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We should mention that this latter family of mappings is intimately related to the so-called accretive operators, which play an important role in the theory of evolution equations.

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Furthermore, if condition (1) holds locally, i.e., if each point $x \in D$ has a neighborhood U such that the restriction of T to U is k-pseudo-contractive with (uniform) constant k, then T is said to be *locally k-pseudo-contractive.*

The purpose of this paper is to continue the discussion concerning the strong convergence of the path $t \to z_t$, $0 \leq t < 1$, defined by (4). In fact, we prove for locally pseudo-contractive mappings under condition (3) that the strong $\lim_{t\to 1^-} z_t$ exists and is a fixed point of T. We should also mention that this result appears to be new even in Hilbert spaces. The first results of this nature were established by Browder [2] and Browder and Petryshyn [3], and more recently Bruck et al. [4] proved Theorem 1 for locally nonexpansive mappings. Another result, perhaps more revealing, is Proposition 2 where we prove that the mapping $2I - T$ is globally one-to-one. This fact, by itself, appears to have a significant connotation in the theory of locally pseudo-contractive mappings.

To fix our notation, we will denote the closure and boundary of D by $\overline{D}$ and ∂D respectively, and for $u, v \in X$ we use seg$[u, v]$ to denote the segment $\{tu + (1-t)v : t \in [0, 1]\}$. We will also use $B(x; r)$ and $\overline{B}(x; r)$ to stand for the open ball $\{z \in X : \|x - z\| < r\}$ and the closed ball $\{z \in X : \|x - z\| \leq r\}$ respectively. We denote the distance between the sets A and B by $\operatorname{dist}(A, B)$, i.e.,

$$\operatorname{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}.$$

II. Preliminaries

The main objective of this paper is to extend Theorem 1 of Morales [11]. We begin by stating this result under the following proposition.

Proposition 1 ([11]). *Let X and X^* be uniformly convex Banach spaces, let D be a bounded open subset of X, and let $T : \overline{D} \to X$ be a uniformly continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that*

$$\|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial D. \tag{3}$$

Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying

$$z_t = tT(z_t) + (1-t)z, \tag{4}$$

where the strong $\lim_{t\to 1^-} z_t$ exists, and this limit is a fixed point for T.

As a consequence of the proof of this previous result, the following can easily be derived.

Corollary 1. *Let X and X^* be uniformly convex Banach spaces, let D be a bounded open subset of X, and let $T : \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that* (3) *holds. Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying* (4). *If, in addition, this path $\{z_t : 0 \leq t < 1\}$ satisfies*

$$\operatorname{dist}(\{z_t\}, \partial D) > 0, \tag{$*$}$$

then the strong $\lim_{t\to 1^-} z_t$ exists, and this limit is a fixed point of T.

In view of Corollary 1, we should observe that uniform continuity is essential to prove condition $(*)$. On the other hand, due to the well-known fact that every locally nonexpansive mapping is globally nonexpansive on convex sets, condition $(*)$ can also be shown (see [7]). This fact allows us to derive the following special case, which was obtained earlier by Bruck et al. [4].

Corollary 2. *Let X and X^* be uniformly convex Banach spaces, let D be an open subset of X, and let $T: \overline{D} \to X$ be a continuous mapping which is locally nonexpansive on D. Suppose* (3) *holds for some $z \in D$. Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying* (4). *If, in addition, this path $\{z_t: 0 \leq t < 1\}$ is bounded, then the strong $\lim_{t\to 1^-} z_t$ exists, and this limit is a fixed point of T.*

III. Main result

Now we are ready to extend Proposition 1 by replacing the uniform continuity by mere continuity. In fact, this process will take place by reformulating the original problem into a problem involving locally nonexpansive mappings. However, using this argument, we will lose some properties which will not allow us to use the result of [11].

Theorem 1. *Let X and X^* be uniformly convex Banach spaces, let D be a bounded open set of X, and let $T: \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that*

$$\|z - Tz\| < \|x - Tx\| \quad \textit{for all } x \in \partial D.$$

Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying

$$z_t = tTz_t + (1-t)z,$$

where the strong $\lim_{t\to 1^-} z_t$ exists, and this limit is a fixed point of T.

Before proving this result, we need to introduce some basic facts that will be used in the proof of the main theorem. We begin with a lemma, whose proof can be found in Kirk [7]. It might be worthwhile to mention that the existence of the path $t \to z_t$ was previously established by Kirk and Morales [8] for general Banach spaces. Therefore, it is the strong convergence of this path which is actually at stake.

Lemma 1 (cf. Fact II of [7]). *Let D be an open subset of a Banach space X, and suppose $F: \overline{D} \to X$ is a continuous mapping which is locally strongly accretive on D. Let $u = F(x)$, $x \in D$, and let $S = \operatorname{seg}[u, v]$ such that $\operatorname{seg}[u, v) \subset F(D)$ for some $v \in X$. Then $v \in F(\overline{D})$ and there exists a unique path* (*up to parametrization*) *whose image Γ begins at x, ends at a point $w \in F^{-1}(v)$, and for which $F(\Gamma) = S$. Moreover, the inverse of the restriction of F to Γ is a Lipschitz mapping of S onto Γ.*

Our next lemma can also be found in Kirk [7]. For complete details of its proof, see pages 94 and 97 of [7] respectively.

Lemma 2. *Let X be a Banach space, and let $T: \overline{D} \to X$ be as in Theorem* 1, *satisfying* (3) *for some $z \in D$. Suppose $Fx = 2x - Tx$. Then*:

(i) $\operatorname{seg}[z, Fz] \subset F(D)$.

(ii) *Let $x \in D$ such that $\|x - Tx\| < \rho = \|z - Tz\|/3$.*

Then $B(Fx; \rho) \subset F(D)$.

Proposition 2. *Let X be a Banach space, let D be a connected open subset of X, and let $T: \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on D. Then the mapping $Fx = 2x - Tx$ is globally one-to-one on D.*

Proof. We first observe that F is continuous on $\overline{D}$ and locally strongly accretive on D and, thus, locally expansive on D. This means, for each $x \in D$, there exists a neighborhood U such that for $u, v \in U$

$$\|u - v\| \le \|Fu - Fv\|.$$

Also, as a consequence of Deimling's domain invariance theorem [5, Theorem 3], F maps open subsets of D onto open sets of X. Now we are ready to show that F is a one-to-one mapping on D. To see this, let $y \in F(D)$. Since $F(D)$ is open, there exists $v \in F(D)$ so that $\mathrm{seg}[v, y] \subset F(D)$. Choose $u \in D$ such that $v = Fu$. Then, by Lemma 1, there exists a unique path $\gamma: [0, 1] \to D$ so that $\gamma(0) = u$, $\gamma(1) = x$ for some $x \in F^{-1}(y)$, and for which $F(\gamma([0, 1])) = \mathrm{seg}[v, y]$. Suppose there is $x_1 \in D$ such that $x_1 \neq x$ and $x_1 \in F^{-1}(y)$. Let $B(x_1; \eta) \subset D$ for some $\eta > 0$. Then there exists $\varepsilon > 0$ for which $B(y; \varepsilon) \subset F(B(x_1; \eta))$. On the other hand, due to the continuity of F at x, there exists $\delta > 0$ such that $F(B(x; \delta)) \subset B(y; \varepsilon)$. Now, if we consider the restriction of F to $\widetilde{D} = D \backslash \overline{B}(x; \delta/2)$, it follows that $\mathrm{seg}[v, y] \subset F(\widetilde{D})$. Once again, there exists a (unique) path $\gamma_1: [0, 1] \to \widetilde{D}$ such that $\gamma_1(0) = u$, $\gamma_1(1) = x_2$ for some $x_2 \neq x$, and for which $F(\gamma_1[0, 1]) = \mathrm{seg}[v, y]$. This contradicts the uniqueness of the path γ. This implies $F^{-1}(y) = \{x\}$, and hence F is a homeomorphism from D onto $F(D)$.

Proof of Theorem 1. In view of Proposition 2, the mapping F is not necessarily one-to-one on $\overline{D}$. Therefore, we will redefine the domain of T to assure that F is also invertible on the boundary of its domain. Due to Theorem 2 of [10], we may select $w \in D$ such that

$$\|w - Tw\| < \|z - Tz\|. \tag{5}$$

We now replace D by the open set D_0 defined by

$$D_0 = \{x \in D: \|x - Tx\| < \|z - Tz\|\}.$$

Then $\partial D_0 \subset D$ and

$$\|w - Tw\| < \|x - Tx\| \quad \text{for } x \in \partial D_0.$$

This means the path $t \to w_t$ for which w satisfies (5) exists and is uniquely defined on $[0, 1)$ (see Lemma 3 of [11]). By Lemma 2(i), we know that $\mathrm{seg}[w, Fw] \subset F(D_0)$, and since by Proposition 2 F^{-1} exists on $F(D_0)$, we derive that F^{-1} is nonexpansive on $\mathrm{seg}[w, Fw]$ and

$$\|w - F^{-1}(w)\| \le \|w - F(w)\| < \|x - Tx\|$$

for all $x \in \partial D_0$. Due to the fact $\partial F(D_0) = F(\partial D_0)$, we may say that for each $y \in \partial F(D_0)$, there exists $x \in \partial D_0$ such that $y = Fx$ and

$$\|w - F^{-1}(w)\| < \|y - F^{-1}(y)\|.$$

Consequently, by Corollary 2, there exists a unique path $t \to u_t \in F(D_0)$, $t \in [0, 1)$, satisfying the equation

$$u_t = tF^{-1}(u_t) + (1-t)w$$

where the $\lim_{t\to 1^-} u_t$ exists and is a fixed point of F^{-1}. Due to uniqueness of the path $t \to w_t$, $F^{-1}(u_t) = w_s$ where $s = 1/(2-t)$, and hence the strong $\lim_{t\to 1^-} w_t$ exists. Since this limit exists for every $w \in D_0$ that satisfies (5), we choose a sequence $\{z^n\}$ in D_0 such that $z^n \to z$. For each z^n, the corresponding path can be written as

$$z_t^n = tT(z_t^n) + (1-t)z^n, \qquad t \in [0, 1]. \tag{6}$$

Let $\eta > 0$ such that $B(z; \eta) \subset D$, and let $k \in \mathbb{N}$ such that $z^n \in B(z; \eta/4)$ for all $n \geq k$. From (6) and the fact that each $z_t^n \in D_0$, we have

$$\|z_t^n - z^n\| = \frac{t}{1-t}\|z_t^n - T(z_t^n)\| \leq \frac{t}{1-t}\|z - Tz\|.$$

Then there exists $t_0 \in (0, 1)$ for which $\|z_t^n - z^n\| < \eta/4$ for $t \in [0, t_0]$ and $n \in \mathbb{N}$. This implies that $z_t^n \in B(z; \eta/2)$ for all $t \in [0, t_0]$ and for all $n \geq k$. Since T is globally pseudo-contractive on $B(z; \eta)$, there exists $j \in J(z_t^n - z_t^m)$ so that

$$\begin{aligned}\langle z_t^n - z_t^m, j\rangle &= t\langle Tz_t^n - Tz_t^m, j\rangle + (1-t)\langle z^n - z^m, j\rangle \\ &\leq t\|z_t^n - z_t^m\|^2 + (1-t)\langle z^n - z^m, j\rangle\end{aligned}$$

for n, $m \geq k$ and $t \in [0, t_0]$. Then we may obtain that

$$\|z_t^n - z_t^m\| \leq \|z^n - z^m\| \quad \text{for all } m, n \geq k \text{ and } t \in [0, t_0].$$

This means the sequence $\{z_t^n\}_{n=1}^\infty$ is a convergent sequence for each $t \in [0, t_0]$, say, $\lim_{n\to\infty} z_t^n = \tilde{z}_t$. Therefore, by (6) we have

$$\tilde{z}_t = tT(\tilde{z}_t) + (1-t)z \quad \text{for } t \in [0, t_0].$$

Once again due to uniqueness of the path, $\tilde{z}_t = z_t$ for all t for which $\{z_t^n\}$ is convergent. We now define the set

$$\begin{aligned}E = \{s \in [0, 1]\colon &\|z_t^n - z_t^m\| \leq \|z^n - z^m\| \text{ for all} \\ &t \in [0, s], n, m \geq n_s, \text{ for some } n_s \in \mathbb{N}\}.\end{aligned}$$

Since $t_0 \in E$ and $z_{t_0} \in D$, there exist $\delta > 0$ and $l \in \mathbb{N}$ such that $B(z_{t_0}; \delta) \subset D$ and

$$\|z_{t_0}^n - z_{t_0}\| < \delta/5 \quad \text{for all } n \geq l.$$

Hence there exists $\alpha > 0$ for which

$$z_t^l \in B(z_{t_0}; \delta/4) \quad \text{for all } t \in (t_0 - \alpha, t_0 + \alpha).$$

This implies that $\|z_t^n - z_{t_0}\| < \delta/2$ for all $n \geq l$ and $t \in (t_0 - \alpha, t_0 + \alpha)$. Otherwise, we may choose $j \geq l$ and $t_1 \in (t_0 - \alpha, t_0 + \alpha)$ so that $\|z_{t_1}^j - z_{t_0}\| = \delta/2$. This implies $z_{t_1}^l$, $z_{t_1}^j \in B(t_{t_0}; \delta)$, and since T is pseudo-contractive on $B(z_{t_0}; \delta)$, we obtain

$$\|z_{t_1}^l - z_{t_1}^j\| \leq \|z^l - z^j\| < \delta/5.$$

This is a contradiction, since $\|z_{t_1}^l - z_{t_1}^j\| \geq \delta/4$. Therefore, $(t_0 - \alpha, t_0 + \alpha) \subset E$. Due to the continuity of the path $t \to z_t^n$, we deduce that $t_0 + \alpha \in E$. This means $[0, 1) \subset E$. It remains to show that $1 \in E$. To see this, we first mention that

$$\|z_t^n - T(z_t^n)\| = \left(\frac{1}{t} - 1\right) \|z^n - z_t^n\|.$$

Since D is bounded, there exists $s \in (0, 1)$ such that

$$\|z_t^n - T(z_t^n)\| \le \rho \quad \text{for all } n \ge 1 \text{ and } t \in [s, 1].$$

Therefore, by Lemma 2(ii), $B(F(z_t^n); \rho) \subset F(D)$. We choose $k \in \mathbb{N}$ such that $k \ge n_s$ and $\|z^n - z^m\| < \rho$ for all n, $m \ge k$. Then

$$\|F(z_t^n) - F(z_t^m)\| \le \|z^n - z^m\| \quad \text{for } n, m \ge k \text{ and } t \in [s, 1].$$

Otherwise, we may find i, $j \ge k$ and $r \in (s, 1)$ such that $\|F(z_r^i) - F(z_r^j)\| > \|z^i - z^j\|$, while $\|F(z_s^i) - F(z_s^j)\| \le \|z^i - z^j\|$. Then there exists $t \in [s, r]$ such that

$$\|z^i - z^j\| < \|F(z_t^i) - F(z_t^j)\| < \rho. \tag{7}$$

Hence $\text{seg}[F(z_t^i), F(z_t^j)] \subset F(D)$, and thus $\|z_t^i - z_t^j\| \le \|F(z_t^i) - F(z_t^j)\|$. Since

$$F(z_t^i) - F(z_t^j) = (2 - \tfrac{1}{t})(z_t^i - z_t^j) + (\tfrac{1}{t} - 1)(z^i - z^j),$$

we derive that $\|F(z_t^i) - F(z_t^j)\| \le \|z^i - z^j\|$. This contradicts (7). Therefore,

$$\|z_t^n - z_t^m\| \le \|z^n - z^m\| \quad \text{for } n, m \ge k \text{ and } t \in [s, 1].$$

This implies $1 \in E$, and hence $E = [0, 1]$. This means there exists $n_0 \in \mathbb{N}$ so that

$$\|z_t^n - z_t^m\| \le \|z^n - z^m\| \quad \text{for all } n, m \ge n_0 \text{ and } t \in [0, 1]. \tag{8}$$

In particular, $\{z_1^n\}$ is a convergent sequence, say, to z_1. Then for $\varepsilon > 0$, (8) implies there exists $n \in \mathbb{N}$ such that $\|z_t^n - z_t\| < \varepsilon/3$ for all $t \in [0, 1]$. Also, we may choose $\delta > 0$ satisfying

$$\|z_t^n - z_1^n\| < \varepsilon/3 \quad \text{for all } t \in (1 - \delta, 1].$$

Therefore,

$$\begin{aligned} \|z_t - z_1\| &\le \|z_t - z_t^n\| + \|z_t^n - z_1^n\| + \|z_1^n - z_1\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for $t \in (1 - \delta, 1]$. This completes the proof.

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References

1. F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.
2. ——, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.
3. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575.
4. R. E. Bruck, W. A. Kirk, and S. Reich, *Strong and weak convergence theorems for locally nonexpansive mappings in Banach spaces*, Nonlinear Anal. TMA **6** (1982), 151–155.
5. K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 365–374.
6. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.
7. W. A. Kirk, *A fixed point theorem for local pseudo-contractions in uniformly convex spaces*, Manuscripta Math. **30** (1979), 89–102.
8. W. A. Kirk and C. H. Morales, *On the approximation of fixed points of locally nonexpansive mappings*, Canad. Math. Bull. **24** (1981), 441–445.
9. C. H. Morales, *Pseudo-contractive mappings and the Leray-Schauder boundary condition*, Comment. Math. Univ. Carolin. **20** (1979), 745–756.
10. ——, *On the fixed-point theory for local k-pseudocontractions*, Proc. Amer. Math. Soc. **81** (1981), 71–74.
11. ——, *Strong convergence theorems for pseudo-contractive mappings in Banach space*, Houston J. Math. **16** (1990), 549–557.

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