THE DISCRETE NATURE OF THE PALEY-WIENER SPACES

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ABSTRACT. The Shannon Sampling Theorem suggests that a function with bandwidth \( \pi \) is in some way determined by its samples at the integers. In this work we make this idea precise for the functions in the Paley-Wiener space \( E^p \). For \( p > 1 \), we make a modest contribution, but the basic result is implicit in the classical work of Plancherel and Pólya (1937). For \( 0 < p \leq 1 \), we combine old and new results to arrive at a characterization of \( E^p \) via the discrete Hilbert transform. This indicates that for such entire functions to belong to \( L^p(\mathbb{R}, dx) \), not only is a certain rate of decay required, but also a certain subtle oscillation.

1. Introduction

In this paper we study, for \( 0 < p \), the space \( E^p \) of entire functions \( f \) of finite exponential type \( \tau \) for which
\[
||f||^p_p = \int_{-\infty}^{+\infty} |f(x)|^p dx < +\infty.
\]

\( E^p \) is clearly a subspace of \( L^p(\mathbb{R}, dx) \), so \( ||f||_p \) is a norm for \( 1 \leq p \) and a quasinorm for \( 0 < p < 1 \). Recall that an entire function \( f \) is of exponential type \( \tau \) if \( f(z) = O(e^{(\tau + \varepsilon)|z|}) \) for all \( \varepsilon > 0 \).

For the sequel, we essentially consider \( \tau = \pi \), as the other cases are handled by a change of variables. Henceforth, \( E^p_{\pi} = E^p \). Our definition of \( E^p \) is motivated by a classical theorem of Paley and Wiener.

Theorem 1 (Paley and Wiener). For an entire function \( f \) to belong to \( E^2 \), it is necessary and sufficient that there exist \( \psi \in L^2([−\pi, \pi]) \) such that
\[
f(z) = \int_{-\pi}^{\pi} \psi(t)e^{itz} dt.
\]

Basic facts about entire functions can be found in [1]; in particular, for \( f \) in \( E^p \), \( |f(x)| \to 0 \), as \( |x| \to +\infty \). This allows for the observation that, unlike the \( L^p(\mathbb{R}, dx) \) spaces, the \( E^p \) spaces are nested: \( E^p \subseteq E^q \), if \( 0 < p \leq q \).

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Many facts about $E^p$, $0 < p < 2$, follow from known facts about $E^2$. ($E^2$ is denoted $PW$ by some authors, e.g., [7].)

A brief review of some of these facts: $E^2$ is the isometric image of $L_2([−\pi, \pi])$ under the inverse Fourier transform and is therefore a Hilbert space. Generally speaking, a function whose Fourier transform is supported in an interval is said to be band-limited; such functions are interpreted as signals, with no frequencies outside the “band”. $E^2$ seems to play a significant role in signal processing applications [5]. Central to the $E^2$ theory is the so-called sinc function

$$\text{sinc}(z) = \frac{\sin \pi z}{\pi z}.$$  

Since $\text{sinc}(z - n)$ is the image of $e^{-int}/\sqrt{2\pi}$ under the inverse transform, the collection $\{\text{sinc}(z - n)\}_{n \in Z}$ is an orthonormal basis of $E^2$.

The cardinal series of a function $f$ is

$$f(x) = \sum_{n=-\infty}^{+\infty} f(n) \text{sinc}(x - n).$$

Many facts about the history of the cardinal series and especially its place in communication theory can be found in the comprehensive article of J. R. Higgins [5]. As we will see, for $p > 1$ the sinc functions play the same role as the standard unit vectors in $l_p$. Although the sinc functions do not belong to $E^p$, for $0 < p \leq 1$, they are still central to our results. A bit of notation: In this paper, $l_p$ will denote the space of $p$-summable sequences indexed on the integers. Also, the term samples of a function $f$ will always refer to the sequence $\{f(n)\}_{n \in Z}$.

2. $E^p$ is a quasi-Banach space

Although $E^p$ is clearly a subspace of $L_p(\mathbb{R}, dx)$, it does not seem to have been noticed that $E^p$ is complete, for values of $p$ other than 2. To show that $E^p$ is closed, it suffices to prove that convergence in $E^p$ forces uniform convergence on compact subsets of $\mathbb{C}$ and preserves type. This and more will follow from the following results of Plancherel and Pólya [6].

Theorem 2 (Plancherel and Pólya). Let $p, \tau > 0$ and $f \in E^p$.

(i) For $y \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} |f(x + iy)|^p \, dx \leq e^{p\tau|y|} \int_{-\infty}^{+\infty} |f(x)|^p \, dx.$$  

(ii) There exists a constant $A > 0$, which depends only on $\tau$ and $p$ so that

$$\sum_{n=-\infty}^{+\infty} |f(n)|^p \leq A \int_{-\infty}^{+\infty} |f(x)|^p \, dx.$$  

Let $z_0 = x_0 + iy_0 \in \mathbb{C}$ and denote $f_{z_0}(u) = f(u + z_0)$. If $f \in E^p$, then $f_{z_0}$
also belongs to $E^p$. Applying (i) and (ii) we see that
\[
|f(z_0)|^p = |f_{z_0}(0)|^p 
\leq \sum_{-\infty}^{\infty} |f_{z_0}(k)|^p 
\leq B \int_{-\infty}^{\infty} |f_{z_0}(t)|^p \, dt 
= B \int_{-\infty}^{\infty} |f_{iy_0}(t + x_0)|^p \, dt 
= B \int_{-\infty}^{\infty} |f_{iy_0}(t)|^p \, dt 
= B \int_{-\infty}^{\infty} |f(t + iy_0)|^p \, dt 
\leq Be^{p|\gamma|} \int_{-\infty}^{\infty} |f(t)|^p \, dt.
\]
Consequently, for $f \in E^p$, 
\[
|f(z_0)| \leq Be^{p|\gamma|} \|f\|_p.
\]
From this, we see that if $(f_n)$ is a Cauchy sequence in $E^p$, then it is Cauchy with respect to the topology of uniform convergence on compacta. Consequently, the limit function is not only in $L^p(\mathbb{R})$, but is entire and of exponential type $\pi$. These observations yield the following result.

**Theorem 3.** For $0 < p$, $E^p$ is complete with respect to the $\| \|_p$ quasinorm.

At this point, perhaps it is worthwhile to list some of the bounded operators on $E^p$ which are of natural interest. It is obvious that real translation maps $E^p$ isometrically into itself, and it follows from Theorem 2 that complex translation also maps $E^p$ boundedly into itself. The map $f \mapsto f_c$, where $f_c(z) = f(cz)$, is a bounded map into $E^p$ for $|c| \leq 1$, but in general $f_c$ may not belong to $E^p$, for $|c| > 1$. Also, it follows from the work of Plancherel and Pólya in [6] that differentiation is a bounded operator from $E^p$ into itself. Finally, it should be observed that part (i) of Theorem 2 implies that the map $f \mapsto e^{i\pi z} f$ is an isometry from $E^p$ into $H^p$ of the upper half-plane.

3. $E^p$ IS ISOMORPHIC TO $l_p$, $p > 1$

At the heart of our subsequent results is the following classical result of Plancherel and Pólya [6].

**Theorem 4 (Plancherel and Pólya).** Let $p, \tau > 0$ and let $f \in E^p_\tau$.

(i) If $\tau < \pi$, then there exists a constant $B > 0$ which depends only on $\tau$ and $p$ so that
\[
\int_{-\infty}^{+\infty} |f(x)|^p \, dx \leq B \sum_{n=-\infty}^{\infty} |f(m)|^p.
\]
(ii) If $\lim_{z \to \infty} f(z)e^{-\pi |z|} = 0$ and if $1 < p$, then (i) holds and the constant $B$ depends only on $p$. 
Now Plancherel and Pólya proved that if a function $f$ of exponential type $\tau$ also belongs to $L_p(\mathbb{R})$, then $f$ satisfies $\lim_{z \to \infty} f(z)e^{-\tau|z|} = 0$. Theorems 2 and 4 allowed them to make the following observation.

**Corollary 1.** Let $1 < p$. There exist constants $C_1$ and $C_2$ (depending only on $p$) such that for all functions in $E^p$

$$C_1 \sum_{n=-\infty}^{+\infty} |f(n)|^p \leq \int_{-\infty}^{+\infty} |f(x)|^p \, dx \leq C_2 \sum_{n=-\infty}^{+\infty} |f(n)|^p.$$

Plancherel and Pólya used this fact to prove that for a function $f$ in $E^p$, $p > 1$, the cardinal series

$$\sum_{n=-\infty}^{\infty} f(n) \text{sinc}(z - n)$$

converges *en moyenne d'ordre* $p$ vers $f$ [6]. Conversely, it is clear from the corollary and Theorem 3 that given a sequence $\{\alpha_n\}$ in $l_p$ one can show that the resulting cardinal series

$$F(z) = \sum_{n=-\infty}^{\infty} a_n \text{sinc}(z - n)$$

represents a unique function in $E^p$, with samples $\{\alpha_n\}$. It is also evident that the sinc functions form a basis, in fact, an unconditional basis for $E^p$, $p > 1$. These remarks, together with Corollary 1, yield the following result.

**Theorem 5.** Let $p > 1$. $E^p$ is isomorphic to $l_p$ via the mapping $f \to \{f(n)\}_{n \in \mathbb{Z}}$.

This result has an obvious consequence.

**Corollary 2.** For $p > 1$, $(E^p)^* \cong E^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We see that the action of a linear functional $\phi \in (E^p)^*$, $\phi \sim g \in E^q$, is given by

$$\phi(f) = \sum_{n \in \mathbb{Z}} f(n)\overline{g(n)} = \int_{-\infty}^{+\infty} f(x)\overline{g(x)} \, dx$$

for all $f \in E^p$.

4. *$E^p$ is isomorphic to the discrete Hardy space $H^p(\mathbb{Z})$, $0 < p \leq 1$*

In many situations, $p = 1$ is a critical value. Recall from the Hardy space theory that since the Hilbert transform is bounded on $L_p$, $p > 1$, $H^p$ and $L_p$ are essentially the same. However, for $0 < p \leq 1$, $H^p$ turns out to be a proper closed subspace of $L_p$ consisting of functions which not only satisfy the size condition but also possess a certain type of cancellation. Indeed, $H^p$ consists of those functions in $L_p$ for which the Hilbert transform (viz., the conjugation operator) is bounded (e.g., see [4] or [2]).

We have found that a parallel situation exists for $E^p$ with respect to $l_p$. Now $l_p = L_p(\mathbb{Z}, d\sigma)$ for $\sigma =$ counting measure; so, by analogy with the standard theory (and at the risk of overusing the Hardy space notation and terminology),
we define the discrete Hardy space, $H^p(Z)$, $0 < p < \infty$, to consist of those sequences $\alpha = \{\alpha_k\} \in l_p$, which satisfy

$$\sum_{k \in Z} \left| \sum_{n \neq k} \frac{\alpha_n}{k - n} \right|^p < +\infty.$$ 

Thus $H^p(Z)$ is the subspace of $l_p$ consisting of those sequences $\alpha = \{\alpha_n\}$ for which the discrete Hilbert transform also belongs to $l_p$.

The discrete Hilbert transform, $H$, of a sequence $\alpha = \{\alpha_n\}$ is defined by

$$H(\alpha)(k) = \sum_{n \neq k} \frac{\alpha_n}{k - n}.$$ 

Note that for any noninteger $c \in \mathbb{R}$, $H_c(\alpha)(k) = \sum_{n \in Z} \alpha_n / (k - n + c)$ yields the same class of sequences. $H_c(\alpha)$ is convolution of the sequence $\alpha$ with the kernel $1/(n + c)$. For the sequel, we will use $H_c$ with $c = \frac{1}{2}$, which we will (by a small abuse of notation) denote by $H$.

$H^1(Z)$ is mentioned by Coifman and Weiss [2, p. 622] as an example of a Hardy space, $H^p(X)$, associated with a space $X$ of homogeneous type; these spaces are the result of extending the atomic decomposition theory for the classical Hardy spaces to more general settings. A caveat regarding notation: $H^p(X)$ is defined atomically in [2]; thus it is not obvious that $H^p(Z)$ as defined above coincides with the corresponding atomic space of the same label in [2], although it is easily seen to contain the atomic space. (Coifman and Weiss suggest that the two are the same [2]; we shall consider the connection in a later paper.)

A priori, it would appear that, for $p > 1$, $H^p(Z)$ and $l_p$ are different. However, Plancherel and Pólya [6] proved that if a sequence $\alpha = \{\alpha_n\} \in l_p$ for $p > 1$, then there is a constant $C > 0$, so that

$$||H(\alpha)||_l_p \leq C||\alpha||_l_p.$$ 

(A discrete version of the M. Riesz theorem.) Thus, for $p > 1$, $H^p(Z)$ and $l_p$ coincide.

For $0 < p \leq 1$, we define an obvious quasinorm on $H^p(Z)$. For $\alpha = \{\alpha_n\}_{n \in Z}$,

$$||\alpha||_{H^p} = ||\alpha||_{l_p} + ||H(\alpha)||_{l_p}.$$ 

That $H^p(Z)$ is complete with respect to this quasinorm will follow from subsequent results.

We recall Plancherel and Pólya's inequality for functions of type strictly less than $\pi$.

$$\int_{-\infty}^{+\infty} |f(x)|^p dx \leq B \sum_{n = -\infty}^{+\infty} |f(m)|^p.$$ 

As we recall, for $p > 1$ this inequality holds for functions of type equal to $\pi$, provided it is known that the function lies in $E^p$. For $0 < p \leq 1$, the inequality cannot hold in general for functions of type equal to $\pi$, even for functions belonging to $E^p$. For example, let

$$g_n(z) = \frac{n \sin(\pi z)}{\pi z(z - n)}.$$
Now
\[ g_n = -\frac{\sin \pi x}{\pi x} + \frac{\sin \pi x}{\pi(x - n)}, \]
so that for \( x \) between 0 and \( n \), \(|g_n(x)| \geq \left| \frac{\sin \pi x}{\pi x} \right| \). Consequently
\[
\int_{-\infty}^{\infty} |g_n(x)|^p dx \geq \int_0^n \left| \frac{\sin \pi x}{\pi x} \right|^p dx,
\]
so that \( \|g_n\|_p \sim |n|^\frac{1}{p} - 1 \), for \( \frac{1}{2} < p < 1 \) and \( \sim \log |n| \) for \( p = 1 \), even though \( \|\{g_n(k)\}\|_p = 2^\frac{k}{2} \) for all \( n \). Similar examples can be constructed for \( 0 < p \leq \frac{1}{2} \). In particular, Plancherel and Pólya's inequality reveals a way to test if a function of type \( \pi \) belongs to \( E^p \); that is, we need only determine whether the samples \( \{f(\frac{n}{2})\} \) belong to \( l_p \), due to the fact that \( f(\frac{n}{2}) \) is of type \( \frac{n}{2} \). It is this simple observation that allows us to show the connection between \( E^p \) and \( H^p(Z) \).

**Theorem 6.** Let \( 0 < p \leq 1 \). If \( f \) belongs to \( E^p \), then \( \{(-1)^n f(n)\} \) belongs to \( H^p(Z) \). Conversely, if \( \{\alpha_n\} \) belongs to \( H^p(Z) \), there is a unique \( f \in E^p \) such that \( f(n) = (-1)^n \alpha_n \).

**Proof.** Let \( f \in E^p \). \( f \) has a cardinal series representation
\[
f(z) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(z - n).
\]
(Since \( f \) is also in \( E^2 \), the cardinal series converges uniformly on compact subsets of \( C \).)

From Theorem 4(i), \( \{f(\frac{n}{2})\} \in l_p \). For even \( n \), we simply recover the original samples of \( f \), which thus belong to \( l_p \). For odd \( n \), \( n = 2k + 1 \), \( k \in Z \),
\[
f \left( \frac{2k + 1}{2} \right) = \sum_m f(m) \text{sinc} \left( \frac{2k + 1}{2} - m \right) = \sum_m f(m) \frac{\sin \left( \frac{2k+1}{2} - m \right)}{\pi \left( \frac{2k+1}{2} - m \right)}
\]
\[
= \frac{\sin \left( \frac{2k+1}{2} \right) \pi}{\pi} \sum_m \cos(m\pi) f(m) \frac{1}{k - m + \frac{1}{2}}
\]
\[
= \frac{(-1)^k}{\pi} \sum_m (-1)^m f(m) \frac{1}{k - m + \frac{1}{2}} = \frac{(-1)^k}{\pi} H((-1)^m f(m))(k).
\]
Since \( \{f(\frac{2k+1}{2})\} \in l_p \), it follows that the Hilbert transform of \( \{(-1)^n f(n)\} \in l_p \), whereby \( \{((-1)^n f(n)) \} \in H^p(Z) \).

Next suppose \( \{\alpha_n\} \in H^p(Z) \). We form the cardinal series
\[
g(z) = \sum_{n=-\infty}^{\infty} (-1)^n \alpha_n \text{sinc}(z - n).
\]
Since \( \{\alpha_n\} \in l_p \), \( g \) is at least in \( E^2 \), and thus we know that the cardinal series converges uniformly on compacta. Now \( g(\frac{n}{2}) \) is of type \( \frac{n}{2} \); thus we may apply Theorem 4(i). The above calculations with the cardinal series show that the sequence of samples of \( g(\frac{n}{2}) \) at the even integers is \( \{\alpha_k\} \), and the sequence of samples of \( g(\frac{n}{2}) \) at the odd integers is \( \{\frac{(-1)^k}{\pi} H((\alpha_n))(k)\}_{k \in Z} \). Consequently the sequence \( \{g(\frac{n}{2})\} \) belong to \( l_p \), whereby \( g \in E^p \).
This proof shows that we can map \( E^p \) onto \( H^p(Z) \) via the map \( f \to \{(-1)^n f(n)\} \). For a function \( f \) in \( E^p \),

\[
\sum_{n=-\infty}^{\infty} \left| f\left(\frac{n}{2}\right) \right|^p = \sum_{k=-\infty}^{\infty} \left| f(k) \right|^p + \sum_{k=-\infty}^{\infty} \left| f\left(\frac{2k+1}{2}\right) \right|^p
\]

\[
= \sum_{k=-\infty}^{\infty} \left| f(k) \right|^p + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \left| \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(m)}{k-m+\frac{1}{2}} \right|^p.
\]

It follows from Theorem 4 and the above proof that there are constants \( C_1, C_2 > 0 \) so that

\[
C_1 \|((-1)^n f(n))\|_{H^p} \leq \|f\|_p \leq C_2 \|((-1)^n f(n))\|_{H^p}
\]

for all \( f \in E^p \).

Thus we see that the \( H^p(Z) \) quasinorm is equivalent to the \( E^p \) quasinorm, so that the map from \( E^p \) onto \( H^p(Z) \) is continuous, thereby yielding the following result.

**Theorem 7.** For \( 0 < p \leq 1 \), \( E^p \) is isomorphic to \( H^p(Z) \).

5. **Comments**

It is clear that sequences in \( H^p(Z) \), \( 0 < p \leq 1 \), must sum to zero. Consequently, for a function in \( E^p \), \( \sum_{n \in \mathbb{Z}} (-1)^n f(n) = 0 \); this also follows from well-known facts from classical harmonic analysis. For a function \( f \) in \( E^p \), the Fourier transform \( \hat{f} \) is continuous on \( \mathbb{R} \) and 0 off of \( [-\pi, \pi] \). The above summation simply reflects the fact that \( \hat{f}(\pm\pi) = 0 \). In fact, it is the cancellation that distinguishes \( H^p(Z) \) from \( l_p \) and, consequently, essentially what distinguishes \( E^p \) from \( L_p \), for \( 0 < p \leq 1 \). As for \( H^p(Z) \), membership in \( E^p \) requires progressively greater cancellation (actually, oscillation) for progressively smaller values of \( p \). We will further examine this and other properties of \( E^p \) in a subsequent paper [3].

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**References**


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