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PARABOLICS ON THE BOUNDARY OF THE DEFORMATION SPACE OF A KLEINIAN GROUP

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ABSTRACT. We present a condition on a loxodromic element L of a Kleinian group G which guarantees that L cannot be made parabolic on the boundary of the deformation space of G, namely, that the fixed points of L are separated by the limit set of a subgroup F of G which is a finitely generated quasifuchsian group of the first kind. The proof uses the collar theorem for short geodesics in hyperbolic 3-manifolds.

1. INTRODUCTION

In [3], Maskit shows that any loxodromic element L in a function group G which represents a simple loop on $\Omega(G)/G$ can be made parabolic on $\partial T(G)$; that is, there is a $\varphi \in \partial T(G)$ with $\varphi(L)$ parabolic. This was generalized by Ohshika [4] to all geometrically finite G.

We consider here the converse question of which loxodromic elements of G can be made parabolic on the boundary of $T(G)$. For geometrically finite G, a complete answer to this question would give a 'combinatorial' description of all geometrically finite points on $\partial T(G)$.

Using the collar theorem of Brooks and Matelski [1] for short geodesics in hyperbolic 3-manifolds, we show that any loxodromic whose fixed points are separated by the limit set of a finitely generated quasifuchsian group of the first kind cannot be made parabolic on $\partial T(G)$.

2. DEFINITIONS AND PRELIMINARIES

We use [2] as our standard reference for definitions.

By a *quasifuchsian* group, we will mean a finitely generated quasifuchsian group of the first kind.

Given a Kleinian group G, a quasifuchsian subgroup F of G, and a loxodromic $L \in G$, say that L is *separated* by F if the fixed points of L are separated by $\Lambda(F)$. If L is separated by F, then neither fixed point of L lies in $\Lambda(F)$, and so $\langle L \rangle \cap F$ is trivial.

The *deformation space* $T(G)$ of a finitely generated Kleinian group is the set of discrete, faithful representations of G into $\mathrm{PSL}_2(\mathbf{C})$ which are induced

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by quasiconformal maps from $\overline{\mathbf{C}}$ to itself, modulo conjugation. That is, for $\varphi \in T(G)$, there is a quasiconformal map f of $\overline{\mathbf{C}}$ to itself, so that $\varphi(g) = fgf^{-1}$ for all $g \in G$. In particular, if L is separated by F in G, then $\varphi(L)$ is separated by $\varphi(F)$ in $\varphi(G)$ for all $\varphi \in T(G)$.

Given a (primitive) loxodromic L in a Kleinian group G, a *collar* for L in G is a tubular neighborhood of the axis of L in $\mathbf{H}^3$ which is precisely invariant under $\langle L \rangle$ in G.

The collar theorem of Brooks and Matelski gives an explicit formula for the radius of a collar for L in terms of the multiplier of L, provided this multiplier is sufficiently close to 1.

Theorem 2.1 [1]. *Given τ such that $|\sinh(\tau/2)| < 1/\sqrt{6}$, every element L of a Kleinian group G with $\mathrm{tr}(L) = \pm 2\cosh(\tau/2)$ has a collar of radius $r(\tau)$, where $r(\tau)$ is defined by*

$$\sinh(r(\tau)) = \sqrt{\frac{1}{4|\sinh(\tau/2)|^2} - \frac{3}{2}}.$$

In particular, there is an absolute constant $\varepsilon_0 > 0$ and a function $d\colon (0, \varepsilon_0) \to \mathbf{R}$, so that, for any Kleinian group G and any loxodromic $L \in G$ with $|\mathrm{tr}^2(L) - 4| \le 4\varepsilon < 4\varepsilon_0$, there is a collar of radius $d(\varepsilon)$ for L in G. Moreover, $d(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

3. Results

We are ready to prove the main result of this note.

Theorem 3.1. *Let G be a finitely generated Kleinian group, and let F be a quasifuchsian subgroup. There exists $c > 0$, dependent on F but independent of G, so that $|\mathrm{tr}^2(\varphi(L)) - 4| \ge c$ for all $\varphi \in T(G)$ and all loxodromics $L \in G$ separated by F.*

Proof. Assume that there exist $\varphi_k \in T(G)$ so that $|\mathrm{tr}^2(\varphi_k(L)) - 4| \to 0$ as $k \to \infty$.

Let H_k be the convex hull of $\Lambda(\varphi_k(F))$ in $\mathbf{H}^3$. Since the $\varphi_k(F)$ are isomorphic for all k, the area of the pleated hyperbolic surface $S_k = \partial H_k/\varphi_k(F)$ is constant.

Using Theorem 2.1, we can find ε_k, going to ∞ as $k \to \infty$, so that $\varphi_k(L)$ has a collar of radius ε_k in $\varphi_k(G)$.

Let x_k be a point of intersection of the axis of $\varphi_k(L)$ with the boundary of H_k. Since $\langle \varphi_k(L) \rangle \cap \varphi_k(F)$ is trivial, the ball B_k of radius ε_k about x_k is precisely invariant under the identity in $\varphi_k(F)$.

The intersection of B_k with ∂H_k contains a disc D_k of radius ε_k which contains x_k. Since D_k is precisely invariant under the identity in $\varphi_k(F)$, it projects to a disc of radius ε_k on S_k.

As $k \to \infty$, the areas of the D_k go to infinity, and so the areas of the S_k go to ∞. This gives the desired contradiction. □

Corollary 3.2. *Let G be a finitely generated Kleinian group, and let $L \in G$ be a loxodromic separated by a quasifuchsian subgroup of G. Then $\varphi(L)$ is loxodromic for all $\varphi \in \partial T(G)$.*

Quasifuchsian subgroups of Kleinian groups give rise to (not necessarily embedded) incompressible surfaces in the corresponding 3-manifolds. In some sense, this theorem gives a collar theorem about these surfaces, namely, that the complex length of any closed geodesic crossing this surface is bounded from below.

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