UNIFORM HOMEOMORPHISMS
BETWEEN THE UNIT BALLS IN $L_p$ AND $l_p$

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Abstract. Let $T : B(L_p) \to B(l_p)$, $1 \leq p < 2$, be a uniform homeomorphism with modulus of continuity $\delta_T$. It is shown that for any $\gamma$, $0 \leq \gamma < \frac{2-p}{2p}$, there exists $K > 0$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that $\delta_T^{-1}(\delta_T(\varepsilon_n)) \geq K\varepsilon_n|\log \varepsilon_n|^\gamma$ for all $\varepsilon_n$.

It was proved by Mazur [6] that for $p, q \geq 1$ the spaces $L_p$ and $L_q$, $l_q$ are homeomorphic. From this work it also follows that the unit balls $B(L_p)$ and $B(L_q)$, $B(l_q)$ are uniformly homeomorphic. However, in Lindenstrauss [4] and Enflo [2] the nonexistence of a uniform homeomorphism between $L_p$ and $L_q$ was established. Enflo also proved that $L_1$ and $l_1$ are not uniformly homeomorphic (see [3, pp. 30–32]), and in [1] this result is generalized by Bourgain to the case $1 \leq p < 2$. From the argument used in [1] and [3] it also follows that the unit balls in $L_p$ and $l_p$ are not Lipschitz equivalent. This was also known before (see [3, p. 27]).

In order to get more quantitative information about uniform homeomorphisms between $L_p$ and $l_p$ we study the modulus of continuity, $\delta_T(\varepsilon)$, defined by

$$\delta_T(\varepsilon) = \sup\{||T(x_1) - T(x_2)|| : \|x_1 - x_2\| \leq \varepsilon\}.$$ 

In [5] it was proved by the author that for $p = 1$ there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and a $K > 0$ such that

$$(*) \quad \delta_T^{-1}(\delta_T(\varepsilon_n)) \geq K\varepsilon_n|\log \varepsilon_n|^\gamma \quad \text{for all } \varepsilon_n.$$ 

In this paper we will give a similar result for the case $1 \leq p < 2$. More precisely, we have the following.

Theorem 1. Let $T : B(L_p) \to B(l_p)$, $1 \leq p < 2$, be a uniform homeomorphism, and let $\gamma$ be any number satisfying $0 \leq \gamma < \frac{2-p}{2p}$. Then there exist a $K > 0$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$(**) \quad \delta_T^{-1}(\delta_T(\varepsilon_n)) \geq K\varepsilon_n|\log \varepsilon_n|^\gamma \quad \text{for all } \varepsilon_n.$$ 

The main idea of the proof of $(*)$ was to construct a noncompact set of well-separated metric midpoints in $L_1$. These points are mapped on "almost
metric midpoints" in \( l_1 \), and by a simple compactness argument we can find well-separated points in \( L_1 \) such that the distance between their images is much smaller. To prove Theorem 1 we just modify the ideas used in proof of (\(*\)). Instead of using well-separated metric midpoints we use well-separated "almost metric midpoints" in \( L_p \). For the proof of Theorem 1 we need the following lemma. Using our method we get the logarithmic estimate. We do not know what is the best estimate.

Lemma 1. Let \( \alpha, \gamma \) be any numbers with \( 0 \leq \alpha, 0 \leq \gamma \leq 1 \). If \( \lim_{\varepsilon \to 0} \frac{\delta_T(\varepsilon)}{\varepsilon |\log \varepsilon|^\gamma} = 0 \), then there exists a sequence \( \{\varepsilon_n\} \) with \( \varepsilon_n \to 0 \) such that

\[
\delta_T \left( \frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}} \right) \leq \frac{1}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}(1 + \log 2|\log \varepsilon_n|^{-1}) \delta_T(\varepsilon)}
\]

for all \( \varepsilon_n \).

In the proofs of Theorem 1 and Lemma 1 the following properties of \( \delta_T(\varepsilon) \) will be used frequently. The proofs are simple and will be omitted.

(a) There exists a \( K > 0 \) such that \( \delta_T(\varepsilon) \geq K \varepsilon \) for all \( \varepsilon \).

(b) For every integer \( N \) we have \( \delta_T(\frac{s}{N}) \geq \frac{1}{N} \delta_T(s) \).

Proof of Lemma 1. Let \( \delta_T(\varepsilon) = K(\varepsilon)\varepsilon |\log \varepsilon|^\gamma \). Then there exists a sequence \( \{\varepsilon_n\} \) with \( \varepsilon_n \to 0 \) such that

\[
K \left( \frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}} \right) \leq K(\varepsilon_n).
\]

To see this we assume the contrary. Then there exists \( \varepsilon_0 \) such that for all \( \varepsilon \), \( 0 < \varepsilon \leq \varepsilon_0 \), we have \( K(\varepsilon) < K \left( \frac{s}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}} \right) \). Let \( 0 < \varepsilon_1 \leq \varepsilon_0 \), and let \( \varepsilon_n = \frac{\varepsilon_{n-1}}{2} \sqrt{1 + |\log \varepsilon_{n-1}|^{-2\alpha}} \), \( n = 1, 2, 3, \ldots \). Then we get \( 0 < K(\varepsilon_1) < K(\varepsilon_2) < K(\varepsilon_3) < \cdots < K(\varepsilon_N) \). This gives a contradiction since, by assumption, \( K(\varepsilon_n) \to 0 \) when \( N \to \infty \). Now let \( \{\varepsilon_n\}, \varepsilon_n \to 0 \), be a sequence satisfying (1) and let \( s = \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}} \). Since \( \gamma \leq 1 \), we get \( |\log \varepsilon_n|^\gamma (1 + \log 2|\log \varepsilon_n|^{-1}) \leq |\log \varepsilon_n|^\gamma (1 + \log 2|\log \varepsilon_n|^{-1}) \). From this inequality and by (1) we obtain for \( \varepsilon_n \) small enough

\[
\delta_T(\varepsilon_n s_n/2) \leq \varepsilon_n s_n K(\varepsilon_n) |\log \varepsilon_n s_n/2|^{\gamma/2} \leq K(\varepsilon_n) |\log \varepsilon_n/2|^{\gamma/2} \leq \delta_T(\varepsilon_n s_n(1 + \log 2|\log \varepsilon_n|^{-1})
\]

and the lemma is proved.

Proof of Theorem 1. Since \( L_p \) contains \( l_2 \) isometrically, it is enough to prove the theorem for \( T \) a uniform homeomorphism from \( B(l_2) \) into \( B(l_p) \). Given \( \gamma, 0 \leq \gamma < \frac{2 - p}{2p} \), we let \( \alpha \) be any number satisfying \( \frac{\alpha}{2} > \gamma < \frac{1}{p} - \gamma \).

We first assume that for some \( K_1 > 0 \) there exists a sequence \( \{\varepsilon_n\} \) with \( \varepsilon_n \to 0 \) such that \( \delta_T(\varepsilon_n) \geq K_1 \varepsilon_n |\log \varepsilon_n|^\gamma \) for all \( \varepsilon_n \). Since we always can find \( K_2 > 0 \) such that \( \delta_T^{-1}(\varepsilon) \geq K_2 \varepsilon \) for all \( \varepsilon > 0 \), the theorem follows trivially for this case. Now, if we cannot find such a sequence for any \( K_1 > 0 \), then we have

\[
\lim_{\varepsilon \to 0} \frac{\delta_T(\varepsilon)}{\varepsilon |\log \varepsilon|^\gamma} = 0.
\]

Thus, by Lemma 1, we can find a sequence \( \{\varepsilon_n\} \) satisfying the inequality (1).
Let $K_1 > 0$ and $K(e)$ be such that $K_1 e \leq \delta_T(e) = K(e)|\log e|^\gamma \forall e \leq 1$. Let $\varepsilon_n$ be in the sequence, and let $r$ be such that

$$0 < r < 1 - \frac{1}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}(1 + \log 2)|\log \varepsilon_n|^{-1}}.$$  

Then we can find $x = \sum x_i e_i$, $y = \sum y_i e_i$ in $B(l_2)$ supported by a finite number of coordinates and with $\|x - y\| < \varepsilon_n$ such that

$$\|T(x) - T(y)\| > (1 - r)\delta_T(\varepsilon_n).$$

By Lemma 1 and by the assumption of $r$ this inequality implies that $\|x - y\| \geq \varepsilon_n/2$. For any $i$ outside the union of the supports of $x$ and $y$ we let $z_i$ be the almost metric midpoint to $x$, $y$ defined by

$$z_i = \frac{x + y}{2} + \|x - y\| \log \|x - y\|^{-1/2}\varepsilon_i.$$  

Then we have

$$\|z_i - x\| = \|z_i - y\| = \frac{\|x - y\|}{2} \sqrt{1 + |\log \|x - y\||^{-2\alpha}} \leq \frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}},$$  

$$\|z_i - z_i\| = \frac{1}{2} \|x - y\| \log \|x - y\|^{-\alpha} \geq \frac{1}{\sqrt{2}} \left|\frac{\varepsilon_n}{2} \log \varepsilon_n|^{-1/2}\varepsilon_i\right|,$$

and

$$\|T(x) - T(z_i)\| \leq \frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}},$$

and

$$\|T(y) - T(z_i)\| \leq \frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}}.$$

Now, let $A$ be a finite set such that up to a negligible error $T(x)$ and $T(y)$ are supported on $A$. We let $\pi$ denote the coordinate projection. Then we have

$$(1 - r)\delta_T(\varepsilon_n) < \|T(x) - T(y)\| \leq \|T(x) - \pi_A T(z_i)\| + \|T(y) - \pi_A T(z_i)\|$$

and

$$\|T(x) - T(z_i)\|^p = \|\pi_A T(x) - \pi_A T(z_i)\|^p + \|\pi_{N\setminus A} T(x) - \pi_{N\setminus A} T(z_i)\|^p$$

$$\geq \|T(x) - \pi_A T(z_i)\|^p + \|\pi_{N\setminus A} T(z_i)\|^p - r.$$  

Similarly for $x$ replaced by $y$. Thus we get

$$2 \left(\frac{\varepsilon_n}{2} \sqrt{1 + |\log \varepsilon_n|^{-2\alpha}}\right)^p \geq \|T(x) - T(z_i)\|^p + \|T(y) - T(z_i)\|^p$$

$$\geq \|T(x) - \pi_A T(z_i)\|^p + \|T(y) - \pi_A T(z_i)\|^p + 2\|\pi_{N\setminus A} T(z_i)\|^p - 2r$$

$$\geq 2^{-p/2p}(1 - r)^p(\delta_T(\varepsilon_n))^p + 2\|\pi_{N\setminus A} T(z_i)\|^p - 2r.$$

Let $C_n = \inf_{i,j; i \neq j} \|T(z_i) - T(z_j)\|$. Then we have the following.

Claim. 

$$\lim_{n \to \infty} \frac{C_n}{\delta_T(\varepsilon_n)|\log \varepsilon_n|^{-(\alpha + \gamma)}} = 0.$$  

To prove the claim we assume the contrary, i.e., there exists $K > 0$ and an infinite subsequence of \{\varepsilon_n\} such that

$$C_n \geq K \delta_T(\varepsilon_n)|\log \varepsilon_n|^{-(\alpha + \gamma)}.$$
Thus we have

\[(ii) \quad \|T(z_i) - T(z_j)\| \geq K\delta_T(e_n | \log e_n|^{-(\alpha + \gamma)})\]

for all \(i, j\) and for all \(e_n\) in the subsequence.

Since \(A\) is finite, we have that \(\{\pi_A T(z_j)\}\) is compact. Thus by (ii) and the triangle inequality we get unit vectors \(e_i\) for which

\[\|\pi_{N\setminus A} T(z_i)\| \geq \frac{1}{2} K\delta_T(e_n | \log e_n|^{-(\alpha + \gamma)})\]

This together with (i) gives

\[2 \left( \delta_T \left( \frac{e_n}{2} \sqrt{1 + |\log e_n|^{-2\alpha}} \right) \right)^p \geq 2^{1-p} K^p \delta_T(e_n | \log e_n|^{-(\alpha + \gamma)})^p - 4r.\]

Since this holds for \(r\) arbitrarily small, we get

\[2 \left( \delta_T \left( \frac{e_n}{2} \sqrt{1 + |\log e_n|^{-2\alpha}} \right) \right)^p \geq 2^{1-p} K^p \delta_T(e_n | \log e_n|^{-(\alpha + \gamma)})^p.\]

Using the fact that

\[\delta_T(e_n | \log e_n|^{-(\alpha + \gamma)}) \geq \frac{1}{2} \log e_n|^{-(\alpha + \gamma)} \delta_T(e_n)\]

and Lemma 1 we obtain

\[2^{1-p} \left( \sqrt{1 + |\log e_n|^{-2\alpha} (1 + \log 2|\log e_n|^{-1})} \right)^p \geq 2^{1-2p} K^p |\log e_n|^{-p(\alpha + \gamma)},\]

so we have

\[|\log e_n|^{p(\alpha + \gamma)} \left( \left( \sqrt{1 + |\log e_n|^{-2\alpha} (1 + \log 2|\log e_n|^{-1})} \right)^p - 1 \right) \geq 2^{-p} K^p.\]

Since by assumption of \(\alpha, \gamma\) we have \(p(\alpha + \gamma) - 2\alpha < 0\) and \(p(\alpha + \gamma) - 1 < 0\), this inequality gives a contradiction for \(e_n\) small enough. Thus we have proved that

\[\lim_{n \to \infty} \delta_T(e_n | \log e_n|^{-(\alpha + \gamma)}) = 0.\]

Now assume that for some \(e_0 > 0\) and \(K > 0\) we have

\[\delta_T^{-1}(\delta_T(e_n)) \leq K\varepsilon |\log\varepsilon|^{\gamma} \quad \text{for all } e_n < e_0.\]

Given integer \(N\) there exists \(e' < e_0\) such that for all \(e_n < e'\) we can find \(i, j\) such that

\[\|T(z_i) - T(z_j)\| < \frac{1}{N} \delta_T(e_n | \log e_n|^{-(\alpha + \gamma)}).\]

Thus we have

\[\frac{1}{\delta} e_n |\log e_n|^{-\alpha} < \frac{1}{\sqrt{2}} \|x - y\| |\log x - y\|^{-\alpha} = \|z_i - z_j\|\]

\[\leq \delta_T^{-1} \left( \frac{1}{N} \delta_T(e_n | \log e_n|^{-(\alpha + \gamma)}) \right) \leq \delta_T^{-1} \left( \delta_T \left( \frac{1}{N} e_n | \log e_n|^{-(\alpha + \gamma)} \right) \right)\]

\[\leq K \frac{1}{N} e_n |\log e_n|^{-(\alpha + \gamma)} |\log e_n + \log \frac{1}{N} | \log e_n|^{-(\alpha + \gamma)} |^{\gamma}\]

\[\leq K \frac{2}{N} e_n |\log e_n|^{-(\alpha + \gamma) + \gamma} \quad \text{for } e_n \text{ small enough}.\]
From this we get \( N \leq 16K \), and by taking \( N \) large enough we get a contradiction. Thus there exists \( K > 0 \) and an infinite subsequence
\[
\left\{ e'_n ; e'_n \frac{1}{N} \varepsilon_n \log \varepsilon_n \right\}
\]
such that \( \delta_{T^{-1}}(\delta_T(e_n)) \geq K \varepsilon_n \log \varepsilon_n \) and the theorem is proved.

REFERENCES


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