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ON A CONJECTURE OF RÉVÉSZ

QI-MAN SHAO

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ABSTRACT. Let $\{X_n, n \geq 1\}$ be i.i.d. random variables with $P(X_i = \pm 1) = \frac{1}{2}$. Révész (1990) proved

$$1 \leq \liminf_{n\to\infty} \max_{0\leq j<n} \max_{1\leq k\leq n-j} (2k\log n)^{-1/2}(S_{j+k} - S_j)$$
$$\leq \limsup_{n\to\infty} \max_{0\leq j\leq n} \max_{1\leq k\leq n-j} (2k\log n)^{-1/2}(S_{j+k} - S_j) \leq K \quad \text{a.s.}$$

and conjectured $K = 1$, where $S_n = \sum_{i=1}^n X_i$. In this note we show that Révész's conjecture is true but the conclusion is not valid for general i.i.d. random variables with finite moment generating function.

1. INTRODUCTION

There has been a great amount of work on increments of partial sums for independent, identically or not necessarily identically distributed random variables during the last two decades. One can refer to Csörgő and Révész [2], Hanson and Russo [5, 6], Shao [9 -11], and the references therein. Lately, for i.i.d. random variables $\{X_n, n \geq 1\}$ with $P(X_n = \pm 1) = \frac{1}{2}$, Révész [8] studied the limit behavior of the sequence

$$L_n = \max_{0\leq j<n} \max_{1\leq k\leq n-j} k^{-1/2}(S_{j+k} - S_j)$$

and proved

$$1 \leq \liminf_{n\to\infty} \frac{L_n}{(2\log n)^{1/2}} \leq \limsup_{n\to\infty} \frac{L_n}{(2\log n)^{1/2}} = K < \infty \quad \text{a.s.},$$

where the exact value of K is unknown (cf. [8, p. 171]). Révész [8] conjectured that $K = 1$. This conjecture is related with the well-known Darling and Erdős (1956) theorem as well as the law of the iterated logarithm

$$\lim_{n\to\infty} (2\log\log n)^{-1/2} \max_{1\leq k\leq n} k^{-1/2} S_k = 1 \quad \text{a.s.}$$

The aim of this note is to give an affirmative answer to the Révész conjecture. Indeed, we obtain the following more general result, which, in turn, shows that

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the conclusion is not valid for general i.i.d. random variables with mean zero, variance one, and finite moment generating function.

In what follows we will use the following notation: $x^+ = \max(0, x)$, $\log x = \ln\max(x, e)$, where ln is the natural logarithm, and $[x]$ denotes the integer part of x.

Theorem 1. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of i.i.d. random variables with* $EX_n = 0$ *and* $EX_n^2 = 1$. *Put* $S_0 = 0$ *and* $S(n) = \sum_{1\le i\le n} X_i$. *Assume*

$$Ee^{s_0|X_1|} < \infty \quad \text{for some } s_0 > 0. \tag{1.1}$$

Let $\rho(x) = \inf_{t\ge 0} e^{-tx} Ee^{tX_1}$ *be the Chernoff function of* X_1. *Define*

$$\alpha(c) = \sup\{x : \rho(x) \ge e^{-1/c}\}, \qquad \alpha^* = \sup_{0<c<\infty} \frac{c^{1/2}\alpha(c)}{\sqrt{2}}.$$

Then we have

$$\lim_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k} - S_j}{(2k\log n)^{1/2}} = \alpha^* \quad a.s. \tag{1.2}$$

Corollary 1. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of i.i.d. random variables with* $P(X_n = \pm 1) = \frac{1}{2}$. *Then we have*

$$\lim_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k} - S_j}{(2k\log n)^{1/2}} = 1 \quad a.s. \tag{1.3}$$

From Theorem 1 and Lemma 1, in the next section, one can obtain immediately

Corollary 2. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of i.i.d. random variables satisfying* $EX_n = 0$, $EX_n^2 = 1$, *and* $Ee^{s_0|X_1|} < \infty$ *for some* $s_0 > 0$. *Assume that*

$$Ee^{tX_1^{+2}} = \infty \quad \text{for every } t > 0. \tag{1.4}$$

Then

$$\lim_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k} - S_j}{(2k\log n)^{1/2}} = \infty \quad a.s. \tag{1.5}$$

2. Proof

We start with a preliminary lemma.

Lemma 1. *Let* X_1 *be a random variable with* $EX_1 = 0$, $EX_1^2 = 1$, *and* $Ee^{s_0|X_1|} < \infty$ *for some* $s_0 > 0$. *Let* $t_0 = \sup\{t \ge 0 : Ee^{tX_1^{+2}} < \infty\}$, $\rho(x)$ *be the Chernoff function of* X_1, *and* α^* *be as in Theorem* 1. *Then*

$$\alpha^* \ge \max\left(1, \frac{1}{\sqrt{2t_0}}\right). \tag{2.1}$$

Proof. From the proof of Theorem 1 (cf. (2.25)) one can see that $\alpha^* \ge 1$. So it suffices to show that

$$\alpha^* \ge \frac{1}{\sqrt{2t_0}}. \tag{2.2}$$

Obviously, (2.2) is trivial if $\alpha^* = \infty$. When $\alpha^* < \infty$, we have

$$\alpha(c) \le \alpha^* \sqrt{\frac{2}{c}} \quad \text{for any } c > 0.$$

From the definition of $\rho(x)$ it follows that for any $c > 0$ and $0 < \varepsilon < 1$

$$\begin{aligned} e^{-1/c} &\ge \rho\left((1+\varepsilon)\alpha^*\sqrt{\frac{2}{c}}\right) = \inf_{t\ge 0} Ee^{t(X_1-(1+\varepsilon)\alpha^*\sqrt{2/c})} \\ &\ge \inf_{t\ge 0} Ee^{t(X_1-(1+\varepsilon)\alpha^*\sqrt{2/c})} I\left\{X_1 \ge (1+\varepsilon)\alpha^*\sqrt{\frac{2}{c}}\right\} \\ &= P\left(X_1 \ge (1+\varepsilon)\alpha^*\sqrt{\frac{2}{c}}\right) \\ &= P\left(\exp\left(\frac{X_1^{+2}}{2(1+\varepsilon)^3\alpha^{*2}}\right) \ge \exp\left(\frac{1}{(1+\varepsilon)c}\right)\right), \end{aligned} \tag{2.3}$$

which yields immediately

$$E\exp\left(\frac{X_1^{+2}}{2(1+\varepsilon)^3\alpha^{*2}}\right) < \infty.$$

Thus, by the definition of t_0

$$t_0 \ge \frac{1}{2(1+\varepsilon)^3\alpha^{*2}}.$$

This proves (2.2), by the arbitrariness of ε.

We give a general result on the increment of a Wiener process, which is of independent interest.

Theorem 2. *Let $\{W(t),\ t \ge 0\}$ be a standard Wiener process. Then*

$$\limsup_{a\to\infty} \sup_{t>0} \sup_{s>0} \frac{|W(t+s)-W(t)|}{(2s(\log\frac{s+t}{s} + \log\log(a \vee (s+\frac{1}{s}))))^{1/2}} = 1 \quad a.s. \tag{2.4}$$

Proof. From the well-known law of the iterated logarithm it is obvious that the left-hand side of (2.4) is greater than or equal to 1 almost surely. Noting that

$$\sup_{t>0} \sup_{s>0} \frac{|W(t+s)-W(t)|}{(2s(\log\frac{s+t}{s} + \log\log(a \vee (s+\frac{1}{s}))))^{1/2}}$$

is a nonincreasing function of a, we only need to show that

$$P\left(\sup_{t>0} \sup_{s>0} \frac{|W(t+s)-W(t)|}{(2s(\log\frac{s+t}{s} + \log\log(a \vee (s+\frac{1}{s}))))^{1/2}} \ge \theta^2\right) \to 0 \tag{2.5}$$

as $a \to \infty$ for every $\theta > 1$. We have

(2.6)

$$\begin{aligned} &\sup_{t>0} \sup_{s>0} \frac{|W(t+s)-W(t)|}{(2s(\log\frac{s+t}{s} + \log\log(a \vee (s+\frac{1}{s}))))^{1/2}} \\ &\quad\le \sup_{-\infty<j<\infty} \sup_{-\infty<i<\infty} \sup_{\theta^{i-1}\le t\le\theta^i} \sup_{\theta^{j-1}\le s\le\theta^j} \frac{|W(t+s)-W(t)|}{(2s(\log\frac{s+t}{s} + \log\log(a \vee \theta^{|j|})))^{1/2}} \\ &\quad\le \sup_{-\infty<j<\infty} \sup_{j\le i<\infty} \sup_{0<t\le\theta^i} \sup_{0\le s\le\theta^j} \theta^{1/2} \frac{|W(t+s)-W(t)|}{(2\theta^j(\log\theta^{i-j} + \log\log(a \vee \theta^{|j|})))^{1/2}}. \end{aligned}$$

Applying Lemma 1.2.1 of Csörgő and Révész (1981), we get that there is a positive constant K depending only on θ such that for each $-\infty < j \le i < \infty$, $a \ge 1$

$$P\left(\sup_{0<t\le\theta^i}\sup_{0\le s\le\theta^j}\frac{|W(t+s)-W(t)|}{(2\theta^j(\log\theta^{i-j}+\log\log(\alpha\vee\theta^{|j|})))^{1/2}}\ge\theta\right) \tag{2.7}$$

$$\le K\theta^{i-j}\exp(-\theta(\log\theta^{i-j}+\log\log(a\vee\theta^{|j|})))$$

$$\le K\theta^{(\theta-1)(i-j)}(|j|+a)^{-\theta}. \tag{2.8}$$

Now (2.5) follows from (2.6) and (2.8) immediately. This completes the proof of the theorem.

We are now ready to prove our main result.

Proof of Theorem 1. We first prove

$$\liminf_{n\to\infty}\max_{0\le j<n}\max_{1\le k\le n-j}\frac{S_{j+k}-S_j}{(2k\log n)^{1/2}}\ge\alpha^* \quad \text{a.s.} \tag{2.9}$$

We have, for every $c>0$,

$$\liminf_{n\to\infty}\max_{0\le j<n}\max_{1\le k\le n-j}\frac{S_{j+k}-S_j}{(2k\log n)^{1/2}}$$

$$\ge\liminf_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{(2[c\log n]\cdot\log n)^{1/2}} \tag{2.10}$$

$$=\left(\frac{c}{2}\right)^{1/2}\liminf_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{[c\log n]}=\frac{c^{1/2}\alpha(c)}{\sqrt{2}} \quad \text{a.s.}$$

by (1.1) and the Erdős-Rényi law of large numbers (cf. [2, p. 98]). (2.9) follows now from (2.10) immediately.

We next show that

$$\limsup_{n\to\infty}\max_{0\le j<n}\max_{1\le k\le n-j}\frac{S_{j+k}-S_j}{(2k\log n)^{1/2}}\le\alpha^* \quad \text{a.s.} \tag{2.11}$$

which together with (2.9) will imply (1.2).

If $\alpha^*=\infty$, (2.11) holds obviously. So we only need to consider the case of $\alpha^*<\infty$ which, by Lemma 1, also implies $t_0>0$. Let $\{W(t),\ t\ge 0\}$ be a standard Wiener process. Using Theorem 2 and Erdős-Rényi law of large numbers, one has

$$\lim_{n\to\infty}\max_{0\le j\le n-[c\log n]}\max_{c\log n\le k\le n-j}\frac{|W(j+k)-W(j)|}{(2k\log n)^{1/2}}=1 \quad \text{a.s.}, \tag{2.12}$$

$$\lim_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{W(j+[c\log n])-W(j)}{(2[c\log n]\log n)^{1/2}}=1 \quad \text{a.s.} \tag{2.13}$$

for each $c>0$. Hence, for any fixed $0<\varepsilon<\frac{1}{2}$, by (2.12), (2.13), and the well-known Komlós-Major-Tusnàdy [7] strong approximation theorem, there exists a positive $c_1=c_1(\varepsilon)$ such that

$$\limsup_{n\to\infty}\max_{0\le j\le n-[c_1\log n]}\max_{c_1\log n\le k\le n-j}\frac{|S_{j+k}-S_j|}{(2k\log n)^{1/2}}\le 1+\varepsilon \quad \text{a.s.}, \tag{2.14}$$

$$\liminf_{n\to\infty}\max_{0\le j\le n-[c_1\log n]}\frac{S_{j+[c_1\log n]}-S_j}{(2[c_1\log n]\log n)^{1/2}}\ge 1-\varepsilon \quad \text{a.s.} \tag{2.15}$$

For $0 < c < c_1$, using the Cauchy-Schwarz inequality, we obtain

$$\tag{2.16}\begin{aligned}
&\limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le (n-j)\wedge c\log n} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \\
&\quad \le \limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k<(n-j)\wedge c\log n} \frac{\sum_{i=1+j}^{j+k} X_i^+}{(2k\log n)^{1/2}} \\
&\quad \le \limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le (n-j)\wedge c\log n} \frac{(\sum_{i=1+j}^{j+k} X_i^{+2})^{1/2}}{(2\log n)^{1/2}} \\
&\quad \le \left(\frac{c}{2} + \limsup_{n\to\infty} \max_{0\le j\le n-[c\log n]} \frac{\sum_{i=1+j}^{j+[c\log n]}(X_i^{+2} - EX_i^{+2})}{2\log n}\right)^{1/2}.
\end{aligned}$$

Set

$$\tilde{\rho}(x) = \inf_{t\ge 0} e^{-tx} E e^{t(X_i^{+2}-EX_i^{+2})}, \quad \tilde{\alpha}(c) = \sup\{x : \tilde{\rho}(x) \ge e^{-1/c}\}.$$

Since $t_0 > 0$,

$$\tag{2.17} \limsup_{n\to\infty} \max_{0\le j\le n-[c\log n]} \frac{\sum_{i=1+j}^{j+[c\log n]}(X_i^{+2} - EX_i^{+2})}{2\log n} = \frac{c\tilde{\alpha}(c)}{2} \quad \text{a.s.}$$

by the Erdős-Rényi law of large numbers.

From the definition of t_0 it follows that for every $0 < t < t_0$

$$e^{-1/c} \le \tilde{\rho}(\tilde{\alpha}(c)) \le e^{-t\tilde{\alpha}(c)} E e^{t(X_1^{+2}-EX_1^{+2})};$$

that is,

$$c\tilde{\alpha}(c) \le \frac{1}{t} + c\ln(Ee^{t(X_1^{+2}-EX_1^{+2})}).$$

Therefore, we can take $0 < c_2 = c_2(\varepsilon) < c_1$ such that

$$\tag{2.18} \left(\frac{c_2}{2} + \frac{c_2\tilde{\alpha}(c_2)}{2}\right)^{1/2} \le \frac{1}{\sqrt{2t_0}} + \varepsilon \le \alpha^* + \varepsilon.$$

By (2.16)–(2.18), we conclude

$$\tag{2.19} \limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le (n-j)\wedge c_2\log n} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \le \alpha^* + \varepsilon \quad \text{a.s.}$$

We show below that

$$\tag{2.20}\begin{aligned}
&\limsup_{n\to\infty} \max_{0\le j<n} \max_{c_2\log n\le k\le c_1\log n} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \\
&\quad \le (1+3\varepsilon) \sup_{c_2\le c\le 1+c_1} \frac{\sqrt{c}\alpha(c)}{\sqrt{2}} \le (1+3\varepsilon)\alpha^* \quad \text{a.s.}
\end{aligned}$$

Let $\eta > 0$ such that $(1+\eta)(1+2\varepsilon) < 1+3\varepsilon$ and $c_2\eta < 1$. Write $d_l =$

$c_2(1+(l+1)\eta)$, $l \geq -1$. We have

(2.21)

$$\begin{aligned}
&\limsup_{n\to\infty} \max_{0\le j<n} \max_{c_2\log n\le k\le c_1\log n} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \\
&\quad\le \limsup_{m\to\infty} \max_{e^m\le n\le e^{m+1}} \max_{0\le j<n} \max_{c_2\log n\le k\le c_1\log n} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \\
&\quad\le \limsup_{m\to\infty} \max_{0\le j<e^{m+1}} \max_{c_2 m\le k\le c_1(m+1)} \frac{S_{j+k}-S_j}{(2km)^{1/2}} \\
&\quad\le \limsup_{m\to\infty} \max_{0\le j\le e^{m+1}} \max_{0\le l\le (c_1-c_2)/(\eta c_2)} \max_{c_2 m(1+l\eta)\le k\le c_2 m(1+(l+1)\eta)} \frac{S_{j+k}-S_j}{(2km)^{1/2}} \\
&\quad\le \limsup_{m\to\infty} \max_{0\le j<e^{m+1}} \max_{0\le l\le (c_1-c_2)/(\eta c_2)} \max_{1\le k\le md_l} \frac{c_2(1+(l+1)\eta)}{(2c_2(1+l\eta))^{1/2}}\cdot\frac{S_{j+k}-S_j}{m\,d_l} \\
&\quad\le (1+\eta)\limsup_{m\to\infty} \max_{0\le j<e^{m+1}} \max_{0\le l\le (c_1-c_2)/(\eta c_2)} \max_{1\le k\le md_l} \frac{\sqrt{d_l}}{\sqrt{2}}\cdot\frac{S_{j+k}-S_j}{m\,d_l}.
\end{aligned}$$

Since $\rho(x)$ is continuous and strictly decreasing for x with $\rho(x)>0$ (cf. [1]), we have

$$\rho((1+\varepsilon)\alpha(d_l)) < e^{-1/d_l} \quad \text{for any } 0\le l\le (c_1-c_2)/(\eta c_2),$$

and hence, we can take a $\delta>0$ such that

(2.22) $$\rho((1+\varepsilon)\alpha(d_l)) \le e^{-(1+\delta)/d_l} \quad \text{for any } 0\le l\le (c_1-c_2)/(\eta c_2).$$

Applying the well-known Ottaviani maximum inequality, the Chernoff theorem in [1], and (2.22), we arrive at

(2.23)

$$\begin{aligned}
&P\left(\max_{0\le j<e^{m+1}} \max_{0\le l\le (c_1-c_2)/(\eta c_2)} \max_{1\le k\le md_l} \frac{\sqrt{d_l}}{\sqrt{2}}\cdot\frac{S_{j+k}-S_j}{m\,d_l} \right.\\
&\qquad\qquad\qquad \left.\ge (1+2\varepsilon)\sup_{c_2\le c\le 1+c_1}\frac{\sqrt{c}\,\alpha(c)}{\sqrt{2}}\right) \\
&\quad\le 2e^{m+1}\sum_{l=0}^{[(c_1-c_2)/(c_2)]} P\left(\max_{1\le k\le md_l}\frac{\sqrt{d_l}}{\sqrt{2}}\cdot\frac{S_k}{m\,d_l}\ge (1+2\varepsilon)\sup_{c_2\le c\le 1+c_1}\frac{\sqrt{c}\,\alpha(c)}{\sqrt{2}}\right) \\
&\quad\le 2e^{m+1}\sum_{l=0}^{[(c_1-c_2)/(\eta c_2)]} P\left(\frac{\sqrt{d_l}}{\sqrt{2}}\frac{S_{[m\,d_l]}}{m\,d_l}\ge (1+\varepsilon)\sup_{c_2\le c\le 1+c_1}\frac{\sqrt{c}\,\alpha(c)}{\sqrt{2}}\right) \\
&\quad\le 4e^{m+1}\sum_{l=0}^{[(c_1-c_2)/(\eta c_2)]} P(S_{[md_l]}\ge [md_l](1+\varepsilon)\alpha(d_l)) \\
&\quad\le 4e^{m+1}\sum_{l=0}^{[(c_1-c_2)/(\eta c_2)]} (\rho((1+\varepsilon)\alpha(d_l)))^{[md_l]} \\
&\quad\le 4e^{m+1}\sum_{l=0}^{[(c_1-c_2)/(\eta c_2)]} \exp(-(1+\delta)[md_l]/d_l) \le \frac{24c_1}{c_2\eta}\exp\left(\frac{2}{c_2}\right)e^{-\delta m},
\end{aligned}$$

provided that m is sufficiently large. This proves (2.20), by (2.21), (2.23), and the Borel-Cantelli lemma.

Putting (2.14), (2.19), and (2.20) together, we obtain

$$\limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k}-S_j}{(2k\log n)^{1/2}} \le \max(1+\varepsilon, (1+3\varepsilon)\alpha^*). \tag{2.24}$$

On the other hand, a combination of (2.15) with the Erdős-Rényi law of large numbers yields

$$\alpha^* \ge 1-\varepsilon,$$

and hence

$$\alpha^* \ge 1 \tag{2.25}$$

by the arbitrariness of ε. (2.11) now follows from (2.24), (2.25), and the arbitrariness of ε, as desired. The proof of Theorem 1 is now complete.

Proof of Corollary 1. By Theorem 1 and Lemma 1, it suffices to show that

$$\alpha^* \le 1. \tag{2.26}$$

It is known that (cf. [2, p. 98])

$$\alpha(c) = 1 \quad \text{for } 0 < c \le 1, \tag{2.27}$$

and if $c > 1$, then $\alpha(c)$ is the only solution of the equation

$$(1+\alpha(c))\ln(1+\alpha(c)) + (1-\alpha(c))\ln(1-\alpha(c)) = \frac{2}{c}. \tag{2.28}$$

An elementary calculation yields

$$(1+x)\ln(1+x) + (1-x)\ln(1-x) \ge x^2$$

for each $0 \le x \le 1$, which implies

$$c\alpha^2(c) \le 2 \quad \text{for each } c > 1 \tag{2.29}$$

by (2.28). This proves (2.26), as desired.

Remark 1. From the proof of Theorem 1, one can see that $\alpha^* < \infty$ if $t_0 > 0$.

Remark 2. Corollary 2 tells us that the necessary and actually sufficient condition for

$$\lim_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} (S_{j+k}-S_j)/(2k\log n)^{1/2} < \infty \quad \text{a.s.}$$

is $Ee^{tX_1^{+2}} < \infty$ for some $t > 0$.

Remark 3. It also looks interesting to study the limit behaviour of the sequence

$$K_n = \max_{0\le j<n} \max_{1\le k\le n-j} (S_{j+k}-S_j)\Big/\left(\sum_{l=j+1}^{j+k} X_l^2\right)^{1/2}, \qquad n = 1, 2, \dots.$$

We conjecture that

$$1 \le \lim_{n\to\infty} \frac{K_n}{(2\log n)^{1/2}} = K < \infty \quad \text{a.s.}$$

as long as $\{X_n,\ n\ge 1\}$ are i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 < \infty$.

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Department of Mathematics, National University of Singapore, Singapore 0511
E-mail address: MATSQM@LEONIS.NUS.SG

Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, People's Republic of China