

## PERIODIC SOLUTIONS OF CERTAIN ABSTRACT WAVE EQUATIONS

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**ABSTRACT.** Existence results of periodic solutions of certain abstract, weakly nonlinear, wave equations are given when eigenvalues of linear parts of those equations are incommensurable to the time period.

### 1. INTRODUCTION

Recently, J. Mawhin and K. Ben-Naoum [1] and P. J. McKenna [2] investigated the existence of periodic solutions of one-dimensional wave equations when the ratio between the space length and the period was irrational. Generally, problems of those kinds lead to problems of small divisors, and for this reason, it is very hard to study such problems. On the other hand, we can very easily study these problems for specific irrational numbers of the ratio. These irrational numbers can be nicely characterized in notions of the number theory [1, 2].

The purpose of this paper is to proceed in the direction of these papers for more general wave equations. We shall study the equation

$$(1.1) \quad u_{tt} + Au = \varepsilon f(u, t),$$

where  $A$  is a selfadjoint, unbounded linear operator with eigenvalues  $\lambda_1 < \lambda_2 < \dots \rightarrow \infty$ ,  $\varepsilon$  is a small parameter, and  $f$  is  $T$ -periodic in  $t \in \mathbb{R}$ . By a  $T$ -periodic solution of (1.1) we mean a weak solution as specified below.

In the first part of this paper, we assume  $\lambda_i \neq 0 \quad \forall i = 1, 2, \dots$ . We shall prove the following result under additional assumptions on  $A, f$ .

**Theorem 1.1.** *Assume there exists a constant  $c > 0$  such that*

$$(1.2) \quad \left| \alpha^2 - \frac{m^2}{\lambda_i} \right| \geq \frac{c}{\lambda_i} \quad \forall m \in \mathcal{N}, \quad \forall \lambda_i > 0,$$

where  $\alpha = T/2\pi$ . Then (1.1) has a weak  $T$ -periodic solution for any  $\varepsilon$  small.

Note that condition (1.2) is satisfied provided

$$(1.3) \quad \left| \alpha - \frac{m}{\sqrt{\lambda_i}} \right| \geq \frac{c}{\lambda_i} \quad \forall m \in \mathcal{N}, \quad \forall \lambda_i > 0,$$

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A weak  $T$ -periodic solution of (1.1) is some  $u \in L^2(S^T, Y)$  satisfying

$$(W) \quad \int_0^T \langle u(t), v_{tt}(t) + Av(t) \rangle dt = \varepsilon \int_0^T \langle f(u(t), t), v(t) \rangle dt$$

$$\forall v \in C^2(S^T, X).$$

We mean the integrability in the sense of Bochner [3]. By (2.1) the Hilbert space  $L^2(S^T, Y)$  has the orthogonal basis

$$(2.2) \quad \left\{ \sin m \frac{2\pi t}{T} \cdot u_j, \cos m \frac{2\pi t}{T} \cdot u_j \right\} \subset C^2(S^T, X)$$

$$m = 0, 1, 2, \dots, j = 1, 2, \dots.$$

**Lemma 2.1.** Assume (1.2). Then the equation

$$\int_0^T \langle u(t), v_{tt}(t) + Av(t) \rangle dt = \int_0^T \langle h(t), v(t) \rangle dt \quad \forall v \in C^2(S^T, X)$$

has a unique solution  $u \in L^2(S^T, Y)$  for any  $h \in L^2(S^T, Y)$ . Moreover,

$$\|u\|_{L^2(S^T, Y)} \leq c \cdot \|h\|_{L^2(S^T, Y)}$$

for a constant  $c > 0$ .

*Proof.* By using both the Fourier expansion of  $h$  in the basis (2.2) and the condition (1.2), we immediately obtain the proof. We point out that trigonometric polynomials of (2.2) are dense in  $C^2(S^T, X)$ .

The proof of Theorem 1.1 follows obviously by Lemma 2.1 together with the Banach fixed point theorem.

*Proof of Theorem 1.2.* If (1.2) is false for some  $\alpha \in (K, K + 1)$ ,  $K > 0$ , then for any  $d > 0$  small there exist  $m, \lambda_i > 0$  such that

$$\left| \alpha^2 - \frac{m^2}{\lambda_i} \right| \leq \frac{d}{\lambda_i}.$$

This implies

$$\left| \alpha - \frac{m}{\sqrt{\lambda_i}} \right| \leq \frac{d}{K\lambda_i}.$$

Since  $\alpha \in (K, K + 1)$ , we have  $\frac{m^2}{\lambda_i} < (K + 1)^2 + 1$  for  $d$  small. Thus

$$m \leq \sqrt{((K + 1)^2 + 1)\lambda_i}.$$

Denote by  $\mathcal{M}$  the set of all  $\alpha \in (K, K + 1)$  for which (1.2) does not hold. Then the Lebesgue measure  $\mu(\mathcal{M})$  of  $\mathcal{M}$  satisfies

$$\mu(\mathcal{M}) \leq \sum_{\lambda_i > 0} \frac{2d}{\lambda_i K} \cdot \sqrt{((K + 1)^2 + 1)\lambda_i} = \frac{2d}{K} \sqrt{((K + 1)^2 + 1)} \sum_{\lambda_i > 0} \frac{1}{\sqrt{\lambda_i}}.$$

Since  $d$  is arbitrarily small,  $\mu(\mathcal{M}) = 0$ . The proof is finished.

## 3. RESONANT CASE

Assume the assumptions for  $A$ ,  $f$  from §2 hold, without (n), since we suppose

$$(r) \quad \exists \lambda_k = 0 \quad \text{and} \quad \dim \ker A < \infty.$$

By (W) any weak  $T$ -periodic solution of (1.1) satisfies

$$(3.1) \quad \int_0^T \langle f(u(t), t), u_p \rangle dt = 0 \quad \forall u_p \in \ker A.$$

Then Lemma 2.1 holds as well for this case but only for  $h$  such that

$$\int_0^T \langle h(t), u_p \rangle dt = 0 \quad \forall u_p \in \ker A.$$

Let  $P: Y \rightarrow \ker A$  be the orthogonal projection. Then (1.1) has the form

$$(3.2) \quad \begin{aligned} w_{tt} + Aw &= \varepsilon Qf(w + u_p, \cdot), \\ 0 &= \tilde{P}f(w + u_p, \cdot) \end{aligned}$$

where

$$Q = I - \tilde{P}, \quad \tilde{P}h = \frac{1}{T} \int_0^T Ph(t) dt, \quad w \in \text{im } Q, \quad u_p \in \ker A.$$

Note that these projections are defined for the following reason:

$$\tilde{P}h = 0 \iff \begin{cases} \int_0^T \langle h(t), u_p \rangle dt = 0 \\ \forall u_p \in \ker A. \end{cases}$$

Now, we are able to solve the first equation of (3.2) for  $\varepsilon$  small as in §2. By inserting this solution  $w(\varepsilon, u_p)$  into the second equation of (3.2) we arrive at the bifurcation equation

$$Q(\varepsilon, u_p) = \tilde{P}f(w(\varepsilon, u_p) + u_p, \cdot).$$

Note

$$\begin{aligned} Q(0, u_p) &= \frac{1}{T} \int_0^T Pf(u_p, t) dt, \\ Q(0, \cdot): \ker A &\rightarrow \ker A. \end{aligned}$$

Summing up we obtain

**Theorem 3.1.** *Let (1.2) be satisfied. Assume  $Q(0, \cdot)$  is  $C^1$ -smooth and there exists  $u_{p_0} \in \ker A$  such that*

$$\begin{aligned} Q(0, u_{p_0}) &= 0, \\ DQ(0, u_{p_0}) &\text{ is regular.} \end{aligned}$$

*Then (1.1) has a weak  $T$ -periodic solution for any  $\varepsilon$  small.*

*Proof.* We have to solve  $Q(\varepsilon, u_p) = 0$ . By the assumptions of Theorem 3.1 this equation has a solution near  $u_{p_0}$  for any  $\varepsilon$  small. Indeed, the Leray-Schauder

degree  $\deg(Q(0, \cdot), B_\delta, 0)$  of  $Q(0, \cdot)$  at  $u_{p_0}$  is well defined for  $\delta > 0$  small, since  $Q(0, \cdot)$  is a local isomorphism. Here

$$B_\delta = \{z \in \ker A \mid \langle z - u_{p_0}, z - u_{p_0} \rangle = \delta^2\}.$$

Moreover, this degree is nonzero. The proof is finished.

**Example.** Consider

$$(3.3) \quad \begin{aligned} u_{tt} - u_{xx} - n^2u &= \varepsilon f(u, t), \\ u(t + T, \cdot) &= u(t, \cdot) \quad \forall t, \\ u(t, 0) = u(t, \pi) &= 0 \quad \forall t, \end{aligned}$$

where  $f: \mathbb{R} \times S^T \rightarrow \mathbb{R}$  is  $C^1$ -smooth and globally Lipschitz in  $u$  and  $n \in \mathcal{N}$ . Now, we have for this case

$$\begin{aligned} Au &= -u_{xx} - n^2u, \quad u(0) = u(\pi) = 0, \\ \ker A &= \{\sin nx\}, \quad \lambda_i = i^2 - n^2, \\ X &= \{u \in C^2([0, \pi], \mathbb{R}) \mid u(0) = u(\pi) = 0\}, \quad Y = L^2([0, \pi], \mathbb{R}), \end{aligned}$$

and the condition (1.2) reads as follows:

$$(3.4) \quad \left| \alpha^2 - \frac{m^2}{i^2 - n^2} \right| \geq \frac{c}{i^2 - n^2} \quad \forall m, i \in \mathcal{N}, i > n,$$

for a constant  $c > 0$ . Note that (3.4) is equivalent to

$$(3.5) \quad \left| \frac{i^2 - n^2}{m^2} - \frac{1}{\alpha^2} \right| \geq \frac{c}{\alpha^2 m^2} \quad \forall m, i \in \mathcal{N}, i > n,$$

for a constant  $c > 0$ .

**Theorem 3.2.** *The equation (3.3) has a weak solution, provided (3.5) holds and there is  $z \in \mathbb{R}$  such that*

$$\begin{aligned} \int_0^T \int_0^\pi f(z \cdot \sin nx, t) \sin nx \, dx \, dt &= 0, \\ \int_0^T \int_0^\pi \frac{\partial f}{\partial u}(z \cdot \sin nx, t) \sin^2 nx \, dx \, dt &\neq 0. \end{aligned}$$

*Proof.* We see that for this case

$$\begin{aligned} Pu &= \frac{2}{\pi} \int_0^\pi u(x) \sin nx \, dx \cdot \sin nx, \\ Q(0, u_p) &= \frac{2}{T\pi} \int_0^T \int_0^\pi f(z \cdot \sin nx, t) \sin nx \, dx \, dt \cdot \sin nx, \\ u_p &= z \cdot \sin nx. \end{aligned}$$

Now applying Theorem 3.1, the proof is finished.

In the last part of this section, we find a criterion similar to [1, 2] for  $\alpha$  which ensures that  $\alpha$  will satisfy (3.4)–(3.5), since these conditions are vague. We see (3.5) is equivalent to

$$(3.6) \quad \inf_{i, m \in \mathcal{N}, i > n} |i^2 - n^2 - \omega^2 m^2| > 0,$$

where  $\omega = 1/\alpha$ . We put  $q = i - n > 0$  and obtain from (3.6)

$$\inf_{q, m \in \mathcal{N}} |q^2 + 2qn - \omega^2 m^2| > 0.$$

We compute

$$\begin{aligned} |q^2 + 2qn - \omega^2 m^2| &= m^2 \left| \left( \frac{q}{m} \right)^2 + 2 \frac{q}{m} n - \omega^2 \right| \\ &= m^2 \left| \frac{q}{m} - \gamma_1 \right| \cdot \left| \frac{q}{m} - \gamma_2 \right|, \end{aligned}$$

where

$$\gamma_{1,2} = -n \pm \sqrt{n^2 + \omega^2}$$

are the roots of  $z^2 + 2zn - \omega^2$ . We see  $\gamma_1 > 0$ ,  $\gamma_2 < 0$ . Hence

$$|q^2 + 2qn - \omega^2 m^2| \geq m^2 |\gamma_2| \cdot \left| \gamma_1 - \frac{q}{m} \right|.$$

By using the result of [1, 2] we know that, if  $\gamma_1$  has a bounded continued fraction expansion, then there exists a constant  $c > 0$  satisfying

$$\left| \gamma_1 - \frac{q}{m} \right| \geq \frac{c}{m^2} \quad \forall q, m \in \mathcal{N}.$$

This implies

$$\inf_{q, m \in \mathcal{N}} |q^2 + 2qn - \omega^2 m^2| > 0.$$

Note

$$\gamma_1 = \sqrt{n^2 + \omega^2} - n, \quad \omega = 1/\alpha,$$

i.e.,

$$\alpha = \frac{1}{\sqrt{\gamma_1^2 + 2\gamma_1 n}}.$$

Summarizing we obtain

**Proposition 3.3.** *Assume*

$$\alpha = \frac{1}{\sqrt{\gamma^2 + 2\gamma n}}$$

*for an irrational number  $\gamma > 0$  possessing a bounded continued fraction expansion. Then the number  $\alpha$  satisfies the condition (3.4).*

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