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A COUNTEREXAMPLE TO THE DEFORMATION CONJECTURE FOR UNIFORM TREE LATTICES

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ABSTRACT. Let X be a universal cover of a finite connected graph. A uniform lattice on X is a group acting discretely and cocompactly on X. We provide a counterexample to Bass and Kulkarni's Deformation Conjecture (1990) that a discrete subgroup $F \leq \mathrm{Aut}(X)$ could be deformed, outside some F-invariant subtree, into a uniform lattice.

A *uniform tree* X is, by definition, the universal cover of a finite connected graph Y. A group $\Gamma < \mathrm{Aut}(X)$ is a uniform X-lattice if Γ is discrete (i.e., every vertex stabilizer Γ_x for $x \in VX$ is finite, where VX is the set of all vertices of X) and the quotient graph $\Gamma \backslash X$ is finite. In this case $Y = \Pi \backslash X$, where $\Pi = \Pi_1(Y)$ is a free group acting freely on X. In particular $G \backslash X$ is finite, where $G = \mathrm{Aut}(X)$, and Π is then a discrete cocompact subgroup (i.e., *uniform lattice*) in the locally compact group G.

The basic theory of such uniform three lattices was developed in Bass-Kulkarni [BK]. In that paper they obtained many important results. It was shown there that if X is a locally finite tree and $G = \mathrm{Aut}(X)$, then in order for X to be uniform one needs not only the finiteness of $G \backslash X$ but also a 'unimodularity condition', that the locally compact group G be unimodular; this condition also has a combinational interpretation. When X is uniform they further showed that there is a uniform lattice $\Gamma \leq G$ such that $\Gamma \backslash X = G \backslash X$, that every free uniform lattice is conjugate to a subgroup of Γ, and that every uniform lattice Γ' is conjugate to one commensurable with Γ. Recall that Γ_0 and Γ_1 are said to be commensurable, denoted $\Gamma_0 \sim \Gamma_1$, if the index $[\Gamma_i : \Gamma_0 \cap \Gamma_1]$ is finite for $i = 0, 1$.

In [BK], one finds the Deformation Conjecture. This asserted, roughly, that a discrete subgroup $F \leq G$ could be 'deformed', outside some F-invariant subtree, into a uniform lattice. This conjecture was proved in [BK] whenever $G \backslash X$ is a tree or a loop. In this paper, we provide a counterexample to the Deformation Conjecture.

Let X be a uniform tree, $G = \mathrm{Aut}(X)$.

Definition. Let $F \leq G$ and let $Y \subset X$ be an F-invariant subtree. By a *deformation* of F outside of Y we mean a monomorphism $h: F \to G$ such that

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$h(f)|Y = f|Y$ for each $f \in F$. We call h a deformation of F into $\Gamma \leq G$ if $h(F) \leq \Gamma$.

Deformation Conjecture ([BK, (4.20)]). Given a discrete group $F \leq G$ and an F-invariant subtree Y such that $F\backslash Y$ is finite, we can deform F outside Y into a uniform X-lattice.

In [BK, (4.24)] it was verified whenever X is homogeneous or when $G\backslash X$ is a tree. We show here, by an explicit example with $G\backslash X$ a triangle, that the Deformation Conjecture fails with F a finite cyclic group.

Before giving the example, we shall recall briefly some concepts introduced in [B1] and [BK].

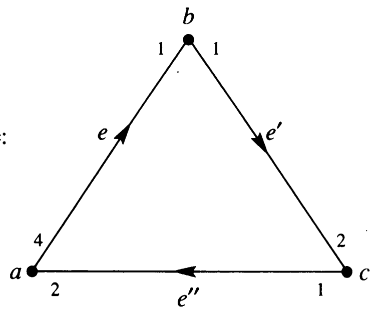
Let X be a uniform tree, Γ be a uniform X-lattice. Suppose that $A = \Gamma\backslash X$ is the quotient of X modulo Γ, $p: X \to A$ is the canonical projection. Choose a subtree $S \subset X$ such that p maps S to A bijectively on edges. Let $i: EA \to \mathbf{Z}_{>0}$ be a function, such that $i(e) = [\Gamma_{\partial_0 \tilde{e}} : \Gamma_{\tilde{e}}]$, where $\Gamma_* = \mathrm{Stab}_\Gamma(*)$, EA is the set of all edges of A, $e \in EA$, $\tilde{e} \in ES$, $p(\tilde{e}) = e$, and $\partial_0 \tilde{e}$ is the initial vertex of $\tilde{e}$. We call (A, i) the edge-indexed graph of X modulo Γ. Taking a base point $a \in A$, then, by [B], the universal cover of (A, i) at a base point $a \in A$ is uniquely determined and is isomorphic to X. Moreover, (A, i) is unimodular, i.e. (see [BK, (1.2), (1.3)]), for any closed edge path $(e_1, e_2, \dots, e_n)$ in A,

$$\prod_{j=1}^{n} i(e_j) / \prod_{j=1}^{n} i(\overline{e_j}) = 1$$

($\partial_0 \overline{e_j} = \partial_1 e_j$, $\partial_1 \overline{e_j} = \partial_0 e_j$, where $\partial_1 e_j$ is the terminal vertex of e_j). Conversely, the universal cover of any unimodular edge-indexed finite graph (A, i) at a fixed base point is a uniform tree. Now we can produce the counterexample.

Example. Consider the edge-indexed graph (cf. [B1, A]):

$(A, i) =:$

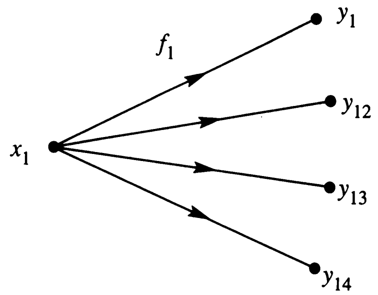


and its universal cover $X = \widetilde{(A, i, a)}$, with projection $p: X \to A$.

Since $\frac{4}{1} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$, (A, i) is unimodular and hence X is a uniform tree. Put $G = \mathrm{Aut}(X)$.

Theorem. *With X as above, there exists a finite cyclic group $F \leq G$ which cannot be deformed outside a finite tree into a uniform X-lattice.*

Proof. Let Y be the following finite subtree of X:

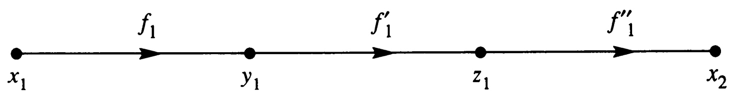


where $p(x_1) = a$ and all edges project to e. Choose $g \in G$ of order 3 with action on Y given by: $g(x_1) = x_1$, $g(y_1) = y_{12}$, $g(y_{12}) = y_{13}$, $g(y_{13}) = y_1$, $g(y_{14}) = y_{14}$. This is possible since $p: X \to A$ maps the four edges of Y to one edge e, i.e., the four edges are in the same G-orbit. Therefore, the four rooted subtrees which are components of $(X - Y)$ connecting to y_1, y_{12}, y_{13}, y_{14} respectively are isomorphic to each other.

Suppose, on the contrary, that F can be deformed outside Y into a uniform X-lattice Γ. Hence, the exists an $\gamma \in \Gamma$ such that $g|Y = \gamma|Y$.

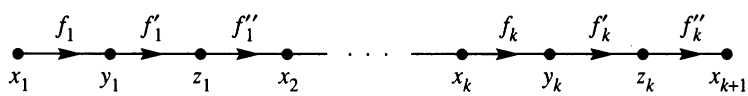
It is shown in [BK, (4.7)] that there is a uniform X-lattice Φ such that $\Phi\backslash X = G\backslash X$. And it is also shown in [BK, (4.15)] that any two uniform lattices are commensurable up to conjugacy. So there exists an $\alpha \in G$ such that Γ and $\alpha\Phi\alpha^{-1}$ are commensurable. Put $\Phi' = \alpha\Phi\alpha^{-1}$; then $\Gamma \sim \Phi'$ and $\Phi'\backslash X = G\backslash X$.

Choose a finite linear subtree $S \subset X$ starting with f_1 such that the canonical projection $p: X \to A = G\backslash X = \Phi'\backslash X$ maps S to $\Phi'\backslash X$ bijectively on edges. Thus S is a three-edge path, say (f_1, f_1', f_1''), with vertex sequence x_1, y_1, z_1, x_2 as follows:



Thus $p(f_1) = e$, $p(f_1') = e'$, $p(f_1'') = e''$, and $p(x_2) = a = p(x_1)$. Choose $\pi \in \Phi'$, such that $x_2 = \pi x_1$.

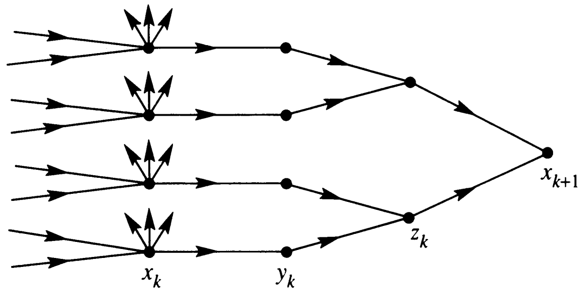
Since $\Gamma \sim \Phi'$, i.e., $[\Phi' : \Gamma \cap \Phi'] < \infty$, there is an integer k, such that $\pi^k \in \Gamma$. Put $f_2 = \pi f_1$. Then (f_1, f_1', f_1'', f_2) is a reduced edge path. Indeed, $p(\overline{f_1''}) = \overline{e''} \neq e = p(f_2)$ implies $\overline{f_1''} \neq f_2$. Inductively, we can deduce that $Z = \bigcup_{i=0}^{k-1} \pi^i S$ is a reduced linear tree:

Z: 

where $f_i = \pi^{i-1} f_1$, $f_i' = \pi^{i-1} f_1'$, $f_i'' = \pi^{i-1} f_1''$, $i = 1, 2, \dots, k$. Thus, $x_{k+1} = \partial_1 f_k'' = \partial_1 \pi^{k-1} f_1'' = \pi^{k-1} \partial_1 f_1'' = \pi^{k-1} x_2 = \pi^k x_1$. As $\pi^k \in \Gamma$,

$$\Gamma_{x_{k+1}} = \Gamma_{\pi^k x_1} = \pi^k \Gamma_{x_1} \pi^{-k}. \tag{1}$$

Since $p(f_i) = p(\pi^{i-1}f_1) = e$, $p(f'_i) = p(\pi^{i-1}f'_1) = e'$, $p(f''_i) = p(\pi^{i-1}f''_1) = e''$, by the indices of (A, i), we have $[G_{x_{i+1}} : G_{f''_i}] = 2$, $[G_{x_i} : G_{f'_i}] = 2$, $[G_{y_i} : G_{f_i}] = 1$, $i = 1, 2, \ldots, k$.



This implies that for each $\alpha \in G_{x_{k+1}}$, α^2 fixes z_k, α^4 fixes y_k and x_k, and, inductively, α^{4^k} fixes x_1. Then for each $\beta \in \Gamma_{x_{k+1}}$,

$$\beta^{4^k} \in \Gamma_{x_1}. \tag{2}$$

Recall that $\gamma \in \Gamma_{x_1}$ has the same action on subtree Y as g does; it follows that

(3) γ^{2^l} does not fix f_1 for any $l \in \mathbf{Z}$ and, hence, does not fix x_{k+1}.

Putting $\gamma_0 = \gamma$, we may inductively define

$$\gamma_i = (\pi^k \gamma_{i-1} \pi^{-k})^{4^k}, \qquad i = 1, 2, 3, \ldots.$$

We claim that for each $j = 1, 2, 3, \ldots$

(a) $\gamma_j = \pi^{jk}\gamma_0^{4^{jk}}\pi^{-jk}$,
(b) $\gamma_j \in \Gamma_{x_1} \cap \Gamma_{x_{k+1}}$ for $j > 0$,
(c) $\gamma_j \neq \gamma_i$, if $j \neq i$.

We prove the claim by induction.

For $j = 1$, by the definition of γ_j, $\gamma_1 = (\pi^k\gamma_0\pi^{-k})^{4^k} = \pi^k\gamma_0^{4^k}\pi^{-k}$, whence (a). Since $\gamma_0 = \gamma \in \Gamma_{x_1}$, it follows from (1) that $\pi^k\gamma_0\pi^{-k} \in \Gamma_{x_{k+1}}$. By (2), $(\pi^k\gamma_0\pi^{-k})^{4^k} \in \Gamma_{x_1}$, whence (b). By (3), $\gamma_0 \notin \Gamma_{x_{k+1}}$, but $\gamma_1 = (\pi^k\gamma_0\pi^{-k})^{4^k} \in \Gamma_{x_{k+1}}$, $\gamma_0 \neq \gamma_1$, whence (c).

Assume that the claim is true for some $j > 0$. Then

(a) $\gamma_{j+1} = (\pi^k\gamma_j\pi^{-k})^{4^k} = \pi^k\gamma_j^{4^k}\pi^{-k} = \pi^k(\pi^{jk}\gamma_0^{4^{jk}}\pi^{-jk})^{4^k}\pi^{-k} = \pi^k\pi^{jk}\gamma_0^{4^{jk}4^k}\pi^{-jk}\pi^{-k} = \pi^{(j+1)k}\gamma_0^{4^{(j+1)k}}\pi^{-(j+1)k}$.

(b) Since $\gamma_{j+1} = (\pi^k\gamma_j\pi^{-k})^{4^k}$ and, by the induction assumption, $\gamma_j \in \Gamma_{x_1} \cap \Gamma_{x_{k+1}}$, it follows, by (1), that $\pi^k\gamma_j\pi^{-k} \in \Gamma_{x_{k+1}}$. Therefore, by (2), $(\pi^k\gamma_j\pi^{-k})^{4^k} \in \Gamma_{x_1} \cap \Gamma_{x_{k+1}}$.

(c) Suppose that $\gamma_m = \gamma_n$ for some $m > n$. Then, by (a), we have

$$\pi^{mk}\gamma_0^{4^{mk}} = \pi^{-mk} = \pi^{nk}\gamma_0^{4^{nk}}\pi^{-nk};$$

then

$$\pi^{(m-n)k}\gamma_0^{4^{mk}}\pi^{-(m-n)k} = \gamma_0^{4^{nk}},$$

i.e., $(\pi^{(m-n)k}\gamma_0^{4(m-n)k}\pi^{-(m-n)k})^{4^{nk}} = \gamma_0^{4^{nk}}$. Then the left side is $\gamma_{m-n}^{4^{nk}}$, $m-n>0$. By (b), $\gamma_{m-n} \in \Gamma_{x_{k+1}}$, so does $\gamma_{m-n}^{4^{nk}}$. But, by (3), the right side $\gamma_0^{4^{nk}} \neq \Gamma_{x_{k+1}}$. Thus we proved the claim.

Therefore, we have infinitely many $\gamma_i \in \Gamma_{x_1}$, $i = 1, 2, \dots$; but Γ_{x_1} is a finite subgroup. This contradiction tells us that $f = \langle g \rangle$ cannot be deformed outside Y into a uniform X-lattice.

We have also prove the following Corollary.

Corollary. *The union of all uniform X-lattices is not dense in G.*

Indeed, in the example above, for $g \in G$ and $Y \subset X$, there is no uniform X-lattice Γ and $\gamma \in \Gamma$, such that $\gamma|Y = g|Y$. This means that g is isolated from the union of all uniform X-lattices.

However, it is proved in [L1] that the commensurability group $C_G(\Gamma) = \{g \in G | g\Gamma g^{-1} \sim \Gamma\}$ is dense in G. More precisely, the group generated by all uniform X-lattices commensurable with Γ is dense in G. (See [L2].)

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