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REAL ANALYTICITY OF HOMEOMORPHIC CR MAPPINGS BETWEEN REAL ANALYTIC HYPERSURFACES IN $\mathbf{C}^2$

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ABSTRACT. In this note, we prove a real analyticity result for smooth CR homeomorphisms in $\mathbf{C}^2$.

1. INTRODUCTION

In this paper we prove the following main result.

Theorem 1. *Let M_1 and M_2 be smooth real analytic hypersurfaces in $\mathbf{C}^2$ with M_1 not Levi falt. If $f: M_1 \to M_2$ is a C^∞ smooth homeomorphic CR mapping that extends holomorphically to one side of M_1, then f extends holomorphically to a neighborhood of M_1 in $\mathbf{C}^2$. Furthermore, if M_1 is of infinite type (resp. finite type) at $p \in M_1$, then M_2 is also of infinite type (resp. finite type) at $f(p)$.*

For M_1 and M_2 pseudoconvex and of finite type, many more general theorems of this type were proved in Bell [B] and Bell and Catlin [BC]. Our result is a consequence of a general reflection principle of Baouendi and Rothschild [BR2] and a unique continuation theorem for holomorphic mappings that we shall prove. We first state the general reflection principle of Baouendi and Rothschild [BR2].

Theorem 2 (Baouendi-Rothschild). *Let M_1 and M_2 be smooth real analytic hypersurfaces in $\mathbf{C}^2$. Let $f: M_1 \to M_2$ be a C^∞ CR mapping that extends holomorphically to one side of M_1. If M_1 is not Levi flat and the Jacobian determinant of f does not vanish to infinite order at a point $p \in M_1$, then f extends holomorphically to a neighborhood of $p \in M_1$ in $\mathbf{C}^2$.*

As one can see in the proof of Theorem 2 in [BR2], the same result remains true as long as the transversal component of the map does not vanish to infinite order at a point of extendability. From this remark, to prove Theorem 1, it suffices to prove that if M_1 is not Levi flat and $f: M_1 \to M_2$ is a CR homeomorphism, then the Jacobian determinant of f does not vanish to infinite order at every point of M_1. This amounts to proving a unique continuation theorem when M_1 is of infinite type and to using a generalized Hopf Lemma by Baouendi and Rothschild [BR1] when M_1 is of finite type. The following unique continuation result will be proved in this paper.

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Theorem 3. *Let M_1 and M_2 be smooth hypersurfaces in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, and let $f: M_1 \to M_2$ be a continuous CR mapping that extends holomorphically to one side of M_1, say M_1^-. If f vanishes to infinite order at $p \in M_1$ and M_2 contains a complex hypersurface through $f(p)$, then $f(M_1 \cup M_1^-) \subset M_2$.*

This result extends a theorem of Bell and Lempert [BL] in which M_2 was assumed to be a Levi flat hypersurface.

The proof of Theorem 3 is contained in the next section. We prove some invariants under CR homeomorphisms which are needed in the proof of Theorem 1. The proof of Theorem 1 will be given in §3.

2. Unique continuation for CR mappings

To prove our results, we have to prove a unique continuation theorem when M_1 is of infinite type which is slightly more general than Theorem 3.

Theorem 4. *Let M_1 and M_2 be smooth hypersurfaces in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, and let $f: M_1 \to M_2$ be a continuous CR mapping that extends holomorphically to one side of M_1, say M_1^-. If the transversal component of f vanishes to infinite order at $p \in M_1$ and M_2 contains a complex hypersurface through $f(p)$, then $f(M_1 \cup M_1^-) \subset M_2$.*

Remarks. (1) A theorem of this kind was first proved by Bell and Lempert in [BL] where M_2 was assumed to be Levi flat. Their methods led to proving other results on unique continuation for holomorphic mappings.

(2) In [BR1], Baouendi and Rothschild proved a generalized complex Hopf Lemma which has proved very useful and will be used in our proof of Theorem 1 when M_1 is of finite type. To state it we introduce some notation. If H is a smooth CR mapping between hypersurfaces M_1 and M_2 in $\mathbf{C}^n$, we denote by $\operatorname{Jac} H$ the Jocobian determinant of H considered as a mapping from the real manifold M_1 to the real manifold M_2. A hypersurface is said to be minimal at a point if it does not contain a complex hypersurface through the point. For the notion of minimal convexity at a point, see [BR1] for definitions.

(3) We give an example from [BR1] to show that a CR homeomorphism does not have to be a diffeomorphism. Let M be the hypersurface in $\mathbf{C}^2$ given by $\overline{w} = we^{i|z|^2}$. It is easy to check that this equation does define a real analytic hypersurface which is neither of finite type nor Levi flat at 0. The mapping $(z, w) \to (\sqrt{3}z, w^3)$ restricts to a CR self map of M which is a CR homeomorphism. But the transversal component of the map vanishes to third order at 0, and hence the map is not a diffeomorphism. An example of this kind was observed by Bell [B] when M is Levi flat: $(z, w^3): M \to M$ where $M = \{\operatorname{Im} w = 0\}$.

Theorem 5 (Baouendi-Rothschild). *Suppose that M_1 is minimal at p and* $\operatorname{Jac} H \not\equiv 0$. *If M_2 is minimally convex at $H(p)$, then the differential of H at p is nonzero.*

What we really proved in Theorem 1 is that if f vanishes to infinite order, then the transversal component of f is identically zero. On the other hand, it is easy to see that if $\operatorname{Jac} H \not\equiv 0$, then $f(M_1 \cup M_1^-) \not\subset M_2$. Hence, we could restate our theorem as follows.

Theorem 6. *Let $H: M_1 \to M_2$ be a smooth CR mapping that extends holomorphically to one side of M_1. If M_2 is not minimal at $H(p)$ and* $\operatorname{Jac} H \not\equiv 0$, *then the differential of H vanishes to at most finite order at p.*

Now we can see that our result can be regarded as a substitute for the generalized complex Hopf lemma of Baouendi and Rothschild when the hypersurface M_2 is not minimal, i.e., when M_2 contains a complex hypersurface.

To prove Theorem 4, we need a unique continuation theorem of one variable.

Lemma 7. *Let U be the upper half disc in the plane, and let $f(z) = u(z) + iv(z)$ be a function holomorphic in U and continuous up to the real axis ($f(0) = 0$). If $|v(x)| \le |u(x)|$ on the real axis and f vanishes to infinite order at the origin, i.e., $|f(z)| \le C_N |z|^N$ for each positive integer N, then f is identically zero.*

Proof. This lemma is a consequence of a unique continuation lemma proved in [HKMP], which says if $u(x) \ge 0$ on the real axis, then f is identically zero. To prove Lemma 7, we consider the holomorphic function $f^2(z) = u^2(z) - v^2(z) + 2iu(z)v(z)$. If $|v(x)| \le |u(x)|$ on the real axis, then the real part of $f^2(z)$ is nonnegative on the real axis and $f^2(z)$ also vanishes to infinite order at the origin as f does. Hence, we conclude that $f^2(z)$ is zero and so is f. The proof of the lemma is complete.

Proof of Theorem 4. By considering one-dimensional complex slices which cut the hypersurface M_1 transversally, it suffices to prove Theorem 4 in the simplified case that $m = 1$, M_1 is equal to the real axis, $p = 0$, and M_1^- is the upper half disc U in the plane. Hence, from this point forward, we will be studying a holomorphic mapping f on the upper half disc U into $\mathbf{C}^n$ which extends continuously to the real axis and which maps the real axis into M_2. Now by a linear change of coordinates at $f(0)$ for M_2, we may assume that $f(0) = 0$ and M_2 is given near 0 by $\operatorname{Im} z_n = h(\operatorname{Re} z_n, z_1, \dots, z_{n-1})$ where $h(0) = \nabla h(0) = 0$. Since M_2 contains a complex hypersurface through 0, it follows that $h(z_1, \dots, z_{n-1}, 0) \equiv 0$ for all $(z_1, \dots, z_{n-1}, 0)$. After a biholomorphic change of coordinates, we can assume that M_2 is given by the equation $\operatorname{Im} z_n = \operatorname{Re} z_n g(\operatorname{Re} z_n, z_1, \dots, z_{n-1})$ where $g(0) = 0$.

Now let $f(\zeta) = (f_1(\zeta), \dots, f_{n-1}(\zeta), f_n(\zeta))$ be a holomorphic mapping from U to M_2 that maps the real axis to M_2. Then

$$\operatorname{Im} f_n(\zeta) = \operatorname{Re} f_n(\zeta) g(\operatorname{Re} f_n(\zeta), f_1(\zeta), \dots, f_{n-1}(\zeta))$$

when z is real. From this and then for small x near 0, we have

$$|\operatorname{Im} f_n(x)| \le |\operatorname{Re} f_n(x)|.$$

This says that the function $f_n(\zeta)$ satisfies the conditions of Lemma 7. Therefore, $f_n(\zeta) \equiv 0$. This implies that $f(M_1 \cup M_1^-) \subset M_2$. The proof of Theorem 4 is complete.

3. CR INVARIANTS OF A HYPERSURFACE

In this section we prove some results on CR invariants of a hypersurface that are needed in the proof of Theorem 1. However, these results are of interest in their own right. Actually we will prove that for a hypersurface, a point being Levi flat is CR invariant under CR homeomorphisms and so is the number of nonzero eigenvalues of the Levi form at a point under CR diffeomorphisms. Let M be a smooth hypersurface in $\mathbf{C}^n$. Let r be the defining function for M, i.e., $M = \{r = 0\}$ where $dr \neq 0$ on M; and we denote throughout $M^- = \{z \in C^n; r < 0\}$. The Levi form of the real hypersurface M at a point $p \in M$ is defined to be the complex hessian of its defining function acting on the maximal complex tangential space $T_p^C M$ to M. To be precise, the Levi form, L_r, is defined as follows:

$$L_r(v, \overline{v}) = \sum_{ij=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} v_i \overline{v_j},$$

where $v \in T_p^C M$.

A point p in M is said to be Levi flat if the Levi form of M at p vanishes identically. A real hypersurface M is said to be Levi flat if its Levi form vanishes identically at every point.

We prove that a point being Levi flat is CR invariant and so is the number of nonzero eigenvalues of the Levi form.

Theorem 8. *Let M_1 and M_2 be C^2 smooth hypersurfaces in $\mathbf{C}^n$, and let f: $M_1 \to M_2$ be a C^1 smooth CR homeomorphism. If $p \in M_1$ is a Levi flat point of M_1, then $f(p)$ is a Levi flat point of M_2. Furthermore, the number of nonzero eigenvalues of the Levi form of M_1 at a point q is the same as that of M_2 at $f(q)$ if f is further assumed to be a diffeomorphism.*

We also consider holomorphic mappings which map a real hypersurface into a Levi flat hypersurface. Bell and Lempert [BL] have shown that such mappings are regular up to M in a flattened complex normal direction for a Levi flat hypersurface.

Theorem 9. *Let M_1 and M_2 be C^2 smooth real hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^m$, respectively, with M_1 not Levi flat and M_2 Levi flat. Suppose that f: $M_1 \to M_2$ is a C^1 smooth CR mapping that extends holomorphically to one side of M_1, say M_1^-; then we have $F(M_1 \cup M_1^-) \subset M_2$.*

An easy consequence of this result is that Levi flatness is CR invariant. Using this result we can also prove that a hypersurface that is not Levi flat is also invariant under CR homeomorphisms.

Theorem 10. *Let f: $M_1 \to M_2$ be a C^1 smooth CR homeomorphism. If M_1 is not Levi flat, neither is M_2.*

Theorems 8 and 9 will be used in proving Theorem 1. We point out that the proofs of Theorems 8–10 are largely based on an identity which shows that the Levi forms of real hypersurfaces under a CR mapping and the degeneracy of the map can be tied together.

Lemma 12. *Let M_1 and M_2 be C^2 smooth real hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^m$, respectively. Suppose that f: $M_1 \to M_2$ is a C^1 smooth CR mapping and that r and ρ are the defining functions of M_1 and M_2, respectively. Let $z_0 \in M$ with $r_n(z_0) = \frac{\partial}{\partial z_n} r(z_0) \neq 0$. Then for z near z_0 on M_1, we have the following identity:*

$$(1) \qquad L_\rho \circ f(f'(z)v, \overline{f'(z)v}) = \frac{1}{r_n(z)} \left\{ \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_i}{\partial z_n} \right\} L_r(v, \overline{v})$$

where $v \in T_z^C M_1$, $f = (f_1, \dots, f_m)$, *and* $L_\rho \circ f$ *is the Levi form of* M_2 *at* $f(z)$.

Proof. Since $r_n(z) \neq 0$ for z near z_0 on M_1, the complex tangential space has a basis consisting of

$$L_k = r_n \frac{\partial}{\partial z_k} - r_k \frac{\partial}{\partial z_n}$$

for $k = 1, \dots, n-1$. To show (1), it suffices to show that it is true for the basis. Since f is CR, then we have

$$\overline{L}_k f_j(z) = 0 \quad \text{for } k = 1, \dots, n-1, j = 1, \dots, m.$$

It follows from $f\colon M_1 \to M_2$ that we have on M_1

$$\rho \circ f(z) = 0. \tag{2}$$

Taking the complex tangential derivative L_k on (2), it follows that

$$\sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f L_k f_i = 0. \tag{3}$$

Taking $\overline{L_k}$ on (3), we get

$$\sum_{ij}^{m} \frac{\partial^2 \rho}{\partial w_i \partial \overline{w_j}} \circ f (L_k f_i)(\overline{L_k f_j}) + \sum_{i=1}^{m} \frac{\partial}{\partial w_i} \circ f \overline{L}_k L_k f_i = 0. \tag{4}$$

In order to show (1), it remains to rewrite (4). We identify the vector fields L_k with the vectors $v_k = [0, \dots, r_n, \dots, -r_k]$. First we notice that the Levi form acting on L_k has the form

$$L_r(v_k, \overline{v}_k) = r_{\overline{n}}(r_n r_{k\overline{k}} - r_k r_{n\overline{k}}) - r_{\overline{k}}(r_n r_{k\overline{n}} - r_k r_{n\overline{n}}). \tag{5}$$

Secondly we have

$$\begin{aligned} \overline{L_k} L_k f_i &= \left(r_{\overline{n}} \frac{\partial}{\partial \overline{z}_k} - r_{\overline{k}} \frac{\partial}{\partial \overline{z}_n} \right) \left(r_n \frac{\partial f_i}{\partial z_k} - r_k \frac{\partial f_i}{\partial z_n} \right) \\ &= r_{\overline{n}} r_{n\overline{k}} \frac{\partial f_i}{\partial z_k} - r_{\overline{n}} r_{k\overline{k}} \frac{\partial f_i}{\partial z_n} - r_{\overline{k}} r_{n\overline{n}} \frac{\partial f_i}{\partial z_k} + r_{\overline{k}} r_{k\overline{n}} \frac{\partial f_i}{\partial z_n}. \end{aligned} \tag{6}$$

It follows from (6) that

$$\begin{aligned} &\sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f_i \overline{L_k} L_k f_i \\ &= r_{\overline{n}} r_{n\overline{k}} \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_i}{\partial z_k} - r_{\overline{n}} r_{k\overline{k}} \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_i}{\partial z_n} \\ &\quad - r_{\overline{k}} r_{n\overline{n}} \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_k}{\partial z_k} + r_{\overline{k}} r_{k\overline{n}} \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_k}{\partial z_n} \\ &= \frac{1}{r_n} \left\{ \sum_{j=1}^{m} \frac{\partial \rho}{\partial w_j} \circ f \frac{\partial f_j}{\partial z_n} \right\} L_r(L_k, \overline{L}_k), \end{aligned} \tag{7}$$

where we have used (5) and the identity

$$r_n \left\{ \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_i}{\partial z_k} \right\} = r_k \left\{ \sum_{i=1}^{m} \frac{\partial \rho}{\partial w_i} \circ f \frac{\partial f_i}{\partial z_n} \right\},$$

which is a rewritten form of (3). Finally, we note that

$$f'(z)v_k = (L_k f_1, \dots, L_k f_m) \tag{8}$$

where v_k is the vector form of L_k as defined earlier. Combining (4), (7), and (8) gives the proof of Lemma 4.

Proof of Theorem 8. If p is a Levi flat point of M_1, then by Lemma 12 we have $L_\rho \circ f(f'v, \overline{f'v}) = 0$ for all $v \in T_p^C M_1$. Since $f\colon M_1 \to M_2$ is a CR homeomorphism, $f'(p)\colon T_p^C M_1 \to T_{f(p)}^C M_2$ is a nonsingular linear transformation, and hence, the Levi form $L\rho$ of M_2 at $f(p)$ is identically zero. This implies that $f(p)$ is a Levi flat point. To prove the second statement of Theorem 8, we notice that if f is a diffeomorphism, then the constant appearing in front of the Levi form in Lemma 12 is not zero and so the number of nonzero eigenvalues of M_1 at p is equal to that of M_2 at $f(p)$. Actually what we have proven is that the set of eigenvalues of M_1 at p is proportional to that of M_2 at $f(p)$.

Proof of Theorem 9. We pick a point on M_1 such that the Levi form is not zero identically at p and, hence, in a neighborhood, say U, of p in M_1. Let p_0 be a point in U. By a linear change of coordinates at p_0 and $f(p_0)$, respectively, we may assume that $p_0 = 0$ and $f(p_0) = 0$ and that $r = \operatorname{Im} z_n + h(\operatorname{Re} z_n, z')$ and $\rho = \operatorname{Im} w_n + (\operatorname{Re} w_n, w')$ where $h(0) = 0$, $\nabla h(0) = 0$, $g(0) = 0$, $\nabla g(0) = 0$, and z', $w' \in \mathbf{C}^{n-1}$. We see from Lemma 12 that at $p_0 = 0$

$$L_\rho \circ f(f'(0)v, \overline{f'(0)v}) = \frac{\partial f_n}{\partial z_n}(0) L_r(v, \overline{v}) \tag{9}$$

where $v \in T_0^C M_1$, $f = (f_1, \dots, f_n)$, and $L_\rho \circ f$ is the Levi form of M_2 at $f(0) = 0$. If M_2 is Levi flat, then we conclude from the above identity that $\partial f_n(0)/\partial z_n = 0$ and by moving p_0 slightly we may have $\partial f_n(z)/\partial z_n \equiv 0$ for $z \in U$. Therefore, $f_n(z)$ is independent of z_n. To show $f_n(z) \equiv 0$, we consider on U

$$\operatorname{Im} f_n(z') + g(\operatorname{Re} f_n(z'), f_1(z), \dots, f_{n-1}(z)) = 0$$

and take the complex tangential derivative L_k as defined in the proof of Lemma 12 to conclude that $f_n(z) \equiv 0$. This completes the proof of Theorem 9.

Proof of Theorem 10. If M_2 is Levi flat, we choose a point $p \in M_1$ such that M_1 does not contain any complex hypersurface through p and then, by a theorem of Trepeau [T], we have that f extend holomorphically to one side of M_1.

By Theorem 9 we have $f(M_1 \cup M_1^-) \subset M_2$ where M_1^- is one side of M_1. But this contradicts the following fact, which is related to a fact in [P].

Lemma 13. *Let $f\colon M_1 \to M_2$ be a continuous* CR *homeomorphism between C^2 smooth real hypersurfaces that extends holomorphically to one side of M_1, say M_1^-; then we have $f(M_1 \cup M_1^-) \not\subset M_2$.*

Proof. Take a sequence $z_k \to p \in M_1$, $z_k \in M - 1^-$; and consider the sets

$$E_k = \{z \in M_1^- \colon f(z) = f(z_k)\}.$$

If $f(M_1 \cup M_1^-) \subset M_2$, then Rank $f < n$ everywhere and E_k are analytic sets in M_1^- of dimension ≥ 1. Each E_k has no more than one limit on M_1 because f is homeomorphic in M_1. By Shiffman's theorem [S], $\overline{E}_k$ are analytic sets in M_1^-. We have $d(p, \overline{E}_k) \to 0$; and since $f\colon M_1 \to M_2$ is a homeomorphism,

$$\lim d(p', \overline{E}_k) > 0$$

for any other point $p' \neq p$ in M_1 (here we denote by $d(p, \overline{E}_k)$ the distance between p and $\overline{E}_k$). By the continuity principle, f extends holomorphically through the point p. We obtain a contradiction because the restriction of f to M_1 cannot be one-to-one since the rank of f is $n-1$ in a neighborhood of p in $\mathbf{C}^n$.

3. Proof of theorem 1

In this section we give a proof of Theorem 1. From now on we assume that $f\colon M_1 \to M_2$ is a smooth homeomorphic CR mapping between real analytic hypersurfaces in $\mathbf{C}^2$. If M_1 is not Levi flat, i.e., the Levi form of M_1 does not vanish identically on M_1, it is well known that if $p \in M_1$, then either M_1 is of finite type at p or of infinite type. In the case of infinite type, M_1 contains a complex curve through p. By our remark in the introduction, it suffices to prove that the transversal component of f does not vanish to infinite order at any point of M_1. We assume that $f = (f_1, f_2)$ where f_2 is the transversal component of f and $z = (z_1, z_2)$ where z_2 is the complex normal direction of M_1 at p.

Case 1. *M_1 is of finite type at p*. By a theorem of Pincuk [P], since M_1 is minimal at p and $f\colon M_1 \to M_2$ is a homeomorphic CR mapping, then the inverse f^{-1} of f is also CR and $f(p)$ is a minimal point of M_2. If M_2 is minimally convex at $f(p)$ as defined in [BR1], then the generalized Hopf lemma of Baouendi and Rothschild shows $\frac{\partial f_2}{\partial z_2}(p) \neq 0$, and therefore the general reflection principle implies that f extends holomorphically to a neighborhood of p in $\mathbf{C}^2$.

If M_2 is not minimally convex at $f(p)$, by a theorem in [BR1], f^{-1}, as a CR mapping, extends holomorphically to a neighborhood of $f(p)$ in $\mathbf{C}^2$. Hence, the map $f^{-1} \circ f$ is well defined in M_1^-, one side of M_1, and is identity on M_1. By the uniqueness of holomorphic functions, we have $f^{-1} \circ f =$ identity in $M_1^- \cup M_1$. Then f is a diffeomorphism. So Theorem 2 applies and we are done in this case too.

Case 2. *M_1 is of infinite type at p*. If M_1 is of infinite type at p, then by Theorem 10, M_2 cannot be Levi flat. On the other hand, since M_1 is real analytic, there is a complex curve passing through p, and therefore the image of the curve under f is also a complex curve in M_2 passing through $f(p)$ since f is a CR homeomorphism. This shows that M_2 is of infinite type at $f(p)$. If f_2 vanishes to infinite order at p, then by Theorem 4 we must have

$$f(M_1^- \cup M_1) \subset M_2.$$

But this is impossible by Lemma 13. So we have proved that f_2 vanishes to finite order, and therefore Theorem 2 applies to conclude that f extends holomorphically to a neighborhood of p in $\mathbf{C}^2$. Combined, this completes the proof of Theorem 1.

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