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AN UPPER BOUND OF $\sum 1/(a_i \log a_i)$ FOR PRIMITIVE SEQUENCES

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ABSTRACT. A sequence $A = \{a_i\}$ of positive integers is called primitive if no term of the sequence divides any other. Erdös conjectures that, for any primitive sequence A,

$$\sum_{a \le n, a \in A} \frac{1}{a \log a} \le \sum_{p \le n} \frac{1}{p \log p}, \quad \text{for } n > 1,$$

where the sum is over all primes less than or equal to n. We show that

$$\sum_{a \in A} \frac{1}{a \log a} \le e^\gamma < 1.7811,$$

where γ is Euler's constant.

A sequence $A = \{a_i\}$ of positive integers is called primitive if no term of the sequence divides any other. We use the notation from [1]. The m^{th} prime is denoted by p_m and variable primes by p. The degree of a primitive sequence, $d^\circ(A)$, is the supremum of $\Omega(a)$, for $a \in A$, where $\Omega(a)$ is the number of prime factors of a counted with multiplicity. If $A = \{1\}$ or $\varnothing$, then set $d^\circ(A) = 0$. Define $f(A) = \sum_{a \in A} 1/(a \log a)$. If $d^\circ(A) = 0$, set $f(A) = 0$. Erdös [1] conjectures that, for any primitive sequence A,

$$\sum_{a \le n, a \in A} 1/(a \log a) \le \sum_{p \le n} 1/(p \log p), \quad \text{for } n > 1,$$

where the sum is over all primes less than or equal to n. In [1] it is proved that this conjecture is equivalent to $f(A) \le \sum_p 1/(p \log p)$ and that $f(A) \le 1.84$ (n.b. $\sum_p 1/(p \log p) \le 1.63671$).

We will prove the following upper bound.

Theorem. *$f(A) \le e^\gamma < 1.7811$, where γ, Euler's constant, is defined by* $\gamma = \lim_{n \to \infty}(\sum_{i=1}^n 1/i - \log n)$.

We first sketch the argument in [1] which leads to the bound 1.84. For a primitive sequence A and positive integer m, let

$$A_m = \{a \in A : \text{ all prime factors of } a \text{ are } \ge p_m\},$$
$$A'_m = \{a \in A_m : p_m | a\},$$
$$A''_m = \{a/p_m : a \in A'_m\}.$$

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Note that each of these sequences is primitive.

Let F_m be an upper bound for $f(A_m)$. Since A_m is the disjoint union of A_{m+1} and A'_m,

$$f(A_m) = f(A_{m+1}) + f(A'_m) \leq F_{m+1} + f(A'_m).$$

As noted in [1], we can suppose that A is finite so that $d^\circ(A_m)$ is finite. If $d^\circ(A_m) \leq 1$, then

$$f(A'_m) \leq 1/(p_m \log p_m)$$

and

$$f(A_m) \leq F_{m+1} + 1/(p_m \log p_m).$$

If $d^\circ(A_m) > 1$, then

$$f(A'_m) \leq f(A''_m)/p_m.$$

By the inductive assumption, $f(A''_m) \leq F_m$, which implies

$$f(A_m) \leq F_{m+1} + f(A'_m) \leq F_{m+1} + F_m/p_m.$$

Assuming $F_{m+1} + F_m/p_m \leq F_m$, implies $F_m \leq F_{m+1}/(1 - 1/p_m)$.

These inequalities allow us to perform a reverse induction on the F_m; namely, if we can write down a suitable F_{m+1}, then

$$F_m \leq \max(F_{m+1} + 1/(p_m \log p_m),\ F_{m+1}/(1 - 1/p_m)).$$

In [1] it is shown that $F_N \leq 1/(\log N + \log\log N - 0.0072847)$ for $N \geq 100001$. Choosing $N = 2002168$ yields the following values for F_m:

F_m	m
0.34724	7
0.37723	6
0.41514	5
0.48855	4
0.61282	3
0.91923	2
1.83845	1

To improve this bound we note that a primitive sequence A is the union of the primitive subsequence B consisting of the primes in A and the primitive sequence C consisting of the remaining elements of A. In particular, $C_m \cap B_m = \varnothing$. Let G_m and H_m be upper bounds for $f(B_m)$ and $f(C_m)$ respectively, then $F_m = G_m + H_m$. The reverse induction can now be applied to the B_m and C_m separately. If B_m is nonempty, then

$$G_m = G_{m+1} + 1/(p_m \log p_m),$$
$$H_m = H_{m+1}.$$

If C_m is nonempty, then

$$G_m = G_{m+1},$$
$$H_m = H_{m+1}/(1 - 1/p_m).$$

If B_1 were nonempty, then we would obtain $F_1 \leq F_2 + 1/(2\log 2) \leq 1.64058$. So we can assume that B_1 is empty. The importance of this assumption is that for $m \geq 2$, we obtain a larger upper bound by always assuming that C_m is nonempty. To see this, let $J_m = J_{m+1}/(1 - 1/p_m)$, for $m < N$, and

$J_N = 1/(\log N + \log\log N - 0.0072847)$. The J_m are upper bounds for $f(C_m)$. We claim that

$$2\,(J_m/(1-1/p_{m-1}) - J_m) \geq 1/(p_{m-1}\log p_{m-1}).$$

To prove this claim, we require results of Rosser and Schoenfield [2].

Lemma. *We have*

$$\frac{e^{-\gamma}}{\log x}\left(1-\frac{1}{\log^2 x}\right) < \prod_{p\leq x}\left(1-\frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x}\left(1+\frac{1}{2\log^2 x}\right),$$

for $x > 1$, *and*

$$n\log n < p_n < n(\log n + \log\log n + 5/4),$$

for $n > 1$.

Some trivial manipulations show that to prove the claim it is enough to show $J_m \geq 1/(2\log p_{m-1})$. By applying the lemma to the product in the denominator of

$$J_m = J_N \frac{\prod_{1\leq j\leq m-1}(1-1/p_j)}{\prod_{1\leq j<N}(1-1/p_j)}$$

and choosing N large enough, we see that the claim is implied by

$$\prod_{1\leq j\leq m-1}(1-1/p_j) \geq \frac{e^{-\gamma}}{2\log p_{m-1}}.$$

This estimate follows from the lemma. These calculations also show that by choosing N large enough, we can make J_1 smaller than any number larger than e^{γ}.

The claim shows that we can always assume that C_m is nonempty since C_1 nonempty implies $H_1 = H_2/(1-1/2) = 2H_2$, so that in the last step all the $H_m/(1-1/p_{m-1}) - H_m$ are multiplied by 2. Hence, $F_m \leq J_m$ and the theorem is proved.

References

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