A REMARK ON POSITIVE RADIAL SOLUTIONS
OF THE ELLIPTIC EQUATION \( \Delta u + K(|x|)u^{(n+2)/(n-2)} = 0 \) IN \( \mathbb{R}^n \)

YASUHIRO SASAHARA AND KAZUNAGA TANAKA

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Abstract. We consider the following semilinear elliptic equation involving critical Sobolev exponents:

\[ -\Delta u = K(|x|)u^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n, \]
\[ u(x) \to 0 \quad \text{as } |x| \to \infty, \]

where \( n \geq 3, K(r) \in C([0, \infty), \mathbb{R}) \). We prove the existence of a positive radial solution with asymptotic behavior \( C/|x|^{n-2} \) at \(|x| = \infty\) under the conditions (i) \( K(r) > 0 \) for all \( r > 0 \), (ii) \( K(0) = K(\infty) \), and (iii) there exist \( C, \delta > 0 \) such that \( K(r) > K(0) - Cr^\delta \) for small \( r > 0 \) and \( K(r) > K(0) - C'r^{-\delta} \) for large \( r > 0 \).

1. Introduction and the result

In this paper, we consider the existence of positive radial solutions of the following semilinear elliptic equation involving critical Sobolev exponents:

\[ -\Delta u = K(|x|)u^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n, \tag{1.1} \]
\[ u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2} \]

where \( n \geq 3, K(r) \in C([0, \infty), \mathbb{R}) \).

This problem arises in Riemannian geometry and the existence of positive radial solutions is deeply investigated by [N, DN, LL, LN, YY1, YY2]. In particular, [DN] showed the following striking phenomena: Let \( \eta(r) \in C^0([0, \infty), \mathbb{R}) \) be a function which is monotonically nonincreasing in \( r \) and satisfies \( \eta(0) \in (0, 1), \eta(r) \geq 0 \). Then

(i) (1.1)–(1.2) possesses no positive radial solutions for \( K(r) \equiv 1 - \eta(r) \);

(ii) (1.1)–(1.2) possesses infinitely many positive radial solutions for \( K(r) \equiv 1 + \eta(r) \) and all of them have the same asymptotic behavior \( C/|x|^{(n-2)/2} \);
(iii) \((1.1)-(1.2)\) possesses infinitely many positive radial solutions for \(K(r) \equiv 1\) and all of them have the same asymptotic behavior \(C/|x|^{n-2}\).

This phenomena shows that if \(K(0), K(\infty) > 0\) and \(K(0) \neq K(\infty)\), we cannot expect the existence of a positive radial solution with asymptotic behavior \(C/|x|^{n-2}\) in general. In this paper we consider the existence of positive radial solutions with asymptotic behavior \(C/|x|^{n-2}\) in the case

\[(1.3) \quad K(0) = K(\infty) > 0 \quad \text{and} \quad K(r) > 0 \text{ for all } r > 0.\]

Our main result is as follows:

**Theorem 1.** Assume \(K(r) \in C([0, \infty), \mathbb{R})\) satisfies

- (K1) \(K(0) = K(\infty) = \lim_{r \to \infty} K(r) > 0\);
- (K2) \(K(r) > 0\) for all \(r > 0\);
- (K3) there are constants \(\delta, C > 0\) such that
  \[K(r) \geq K(0) - Cr^\delta \quad \text{for small } r,\]
  \[K(r) \geq K(0) - Cr^{-\delta} \quad \text{for large } r.\]

Then \((1.1)-(1.2)\) possesses at least one positive radial solution with asymptotic behavior \(C/|x|^{n-2}\).

We remark that [LL, YY1, YY2] also studied the situations \((1.3)\). In particular, by the minimizing method Lin and Lin [LL] obtained the existence of a positive radial solution with asymptotic behavior \(C/|x|^{n-2}\) in the conditions: \(K(0) > 0, K(r) = K(0) + \beta r^\alpha + o(r^\alpha)\) for small \(r > 0, \int_1^\infty K(r)r^{-n-1} dr < \infty\), and \(K(\infty) \leq K(0)\), where \(\alpha \in (0, n), \beta > 0\). By shooting methods, [YY1, YY2] obtained the existence of a positive radial solution with asymptotic behavior \(C/|x|^{n-2}\) under the conditions: \(K(0) = K(\infty) > 0\) and \(K(r) \geq K(0)\) for all \(r\). We remark that [YY1, YY2] also obtained a sharp result on the structure of positive radial solutions.

To prove our theorem, we introduce a transformation

\[(1.4) \quad v(t) = e^{(n-2)t/2}u(e^t).\]

We see \(u(x) = u(|x|)\) is a positive radial solution with asymptotic behavior \(C/|x|^{n-2}\) if and only if \(v(t)\) is a positive solution of

\[(1.5) \quad -v'' + \left(\frac{n-2}{2}\right)^2 v = K(e^t)v^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}, \quad v \in H^1(\mathbb{R}).\]

We apply the recent remarkable existence result due to Bahri and Li [BYL] and Bahri and Lions [BPL] to \((1.5)\). Finally we remark our transformation \((1.4)\) was used in [DN] (in the proof of Theorem 5.33) to study the asymptotic behavior of solutions.

2. Proof of Theorem 1

We use variational methods to find a positive radial solution of \((1.1)-(1.2)\). Let \(D^{1,2}(\mathbb{R}^n)\) be the completion of \(C_0^\infty(\mathbb{R}^n)\) with respect to the norm

\[\|u\|_{D^{1,2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right)^{1/2}.\]
and we define \( E = \{ u \in D^{1,2}(\mathbb{R}^n) ; u(x) = u(|x|) \} \), i.e., \( u \) is a radial function

and

\[
\|u\|_E = \left( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{1/2}.
\]

We consider the functional

\[
I(u) = \int_{\mathbb{R}^n} \left[ \frac{1}{2} |\nabla u|^2 - \frac{2n}{n-2} K(|x|) |u|^{2n/(n-2)} \right] \, dx \in C^2(E, \mathbb{R}).
\]

It is known that critical points \( u \in E \) with \( u(x) \geq 0 \) \((u(x) \neq 0)\) are positive radial solutions of (1.1)–(1.2). As to the asymptotic behavior of \( u(x) \) as \( |x| \to \infty \), we consider the Kelvin transformation \( \overline{u}(x) \) of \( u(x) \), that is,

\[
\overline{u}(x) = |x|^{-(n-2)} u \left( \frac{x}{|x|^2} \right).
\]

We can verify \( \overline{u}(x) \) is a critical point of

\[
\overline{I}(u) = \int_{\mathbb{R}^n} \left[ \frac{1}{2} |\nabla u|^2 - \frac{2n}{n-2} K(|x|) |u|^{2n/(n-2)} \right] \, dx.
\]

Thus \( \overline{u}(x) \) is a positive solution of the corresponding equation, in particular, \( \overline{u}(x) \in C(\mathbb{R}^n) \). Thus there exists a limit \( \overline{u}(0) = \lim_{|x| \to 0} \overline{u}(x) \). That is, the limit \( \lim_{|x| \to \infty} |x|^{n-2} u(x) \) exists. Therefore, critical points of \( I(u) \) with \( u(x) \geq 0 \) \((u(x) \neq 0)\) correspond to positive radial solutions with asymptotic behavior \( C/|x|^{n-2} \).

For \( u(x) = u(|x|) \in E \), we define a function \((Tu)(t) : \mathbb{R} \to \mathbb{R}\) by

\[
(Tu)(t) = e^{(n-2)t/2} u(e^t).
\]

Then we have

**Proposition 2.** \( T \) is a bounded linear operator with a bounded inverse from \( E \) to \( H^1(\mathbb{R}) \). More precisely, \( T \) satisfies for all \( u \in E \)

\[
\|u\|_E = \omega_{n-1} \left( \|(Tu)_t\|_{L^2(\mathbb{R})}^2 + \left( \frac{n-2}{2} \right)^2 \|Tu\|_{L^2(\mathbb{R})}^2 \right).
\]

Furthermore, \( T \) satisfies for all \( u \in E \)

\[
\int_{\mathbb{R}^n} K(|x|) |u|^{2n/(n-2)} \, dx = \omega_{n-1} \int_{\mathbb{R}} K(e^t) |(Tu)(t)|^{2n/(n-2)} \, dt.
\]

Here we used the notation: \( \omega_{n-1} = \int_{S^{n-1}} d\sigma \), \( S^{n-1} = \{ x \in \mathbb{R}^n ; |x| = 1 \} \), and \( \|u\|_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} |u|^2 \, dx \right)^{1/2} \).

**Proof.** Since \( C_0^\infty(\mathbb{R}^n) \) is dense in \( E \), it suffices to verify (2.2) and (2.3) for \( u \in C_0^\infty(\mathbb{R}^n) \). By the straightforward computation, we can get (2.2), (2.3) easily.

We can also easily verify the following properties of \( T \).

**Proposition 3.** (i) \((Tu_\lambda)(t) = (Tu)(t + \log \lambda)\) for all \( u \in E \), \( \lambda > 0 \), and \( t \), where \( u_\lambda(x) = \lambda^{(n-2)/2} u(\lambda x) \).
(ii) \((T\bar{u})(t) = (Tu)(-t)\) for all \(u \in E\) and \(t\), where \(\bar{u}(x)\) is the Kelvin transformation of \(u(x)\) defined in (2.1).

We introduce a functional \(J(v): H^1(\mathbb{R}) \to \mathbb{R}\) by

\[
J(v) = \int_\mathbb{R} \left[ \frac{1}{2} \left( |v_t|^2 + \left( \frac{n-2}{2} \right) |v|^2 \right) - \frac{2n}{n-2} K(e') |v|^{2n/(n-2)} \right] dt.
\]

By Proposition 2, we can see

\[
I(u) = \omega_{n-1} J(Tu) \quad \text{for all} \quad u \in E.
\]

Thus we have

**Proposition 4.** \(u \in E\) is a critical point of \(I(u)\) if and only if \((Tu)(t) \in H^1(\mathbb{R})\) is a critical point of \(J(v)\), that is, \((Tu)(t)\) is a solution of (1.5).

Recently there has been remarkable progress due to Bahri and Li [BYL] and Bahri and Lions [BPL] in the study of the existence of critical points of the functional:

\[
F(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2}(|\nabla u|^2 + m|u|^2) - \frac{1}{p+1} q(x)|u|^{p+1} \right] dx \in C^2(H^1(\mathbb{R}^N), \mathbb{R}),
\]

where \(m > 0, p \in (1, (N + 2)/(N - 2))\) if \(N \geq 3\), \(p \in (1, \infty)\) if \(N = 1, 2\), and \(q(x) \in L^\infty(\mathbb{R}^N)\). Critical points of \(F(u)\) correspond to solutions of the following semilinear elliptic equations:

\[
-\Delta u + mu = q(x)|u|^{p-1}u \quad \text{in} \ \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N).
\]

Their result is

**Theorem 5** ([BYL, Theorem 3.1], cf. [BPL]). Assume \(q(x) \in L^\infty(\mathbb{R}^N)\) satisfies

\begin{align*}
(\text{Q1}) \quad & q(x) > 0 \quad \text{for all} \quad x; \\
(\text{Q2}) \quad & \lim_{|x| \to \infty} q(x) \equiv q_\infty > 0 \quad \text{exists}; \\
(\text{Q3}) \quad & \text{there exist constants} \ \delta, C > 0 \ \text{such that} \\
& q(x) \geq q_\infty - C \exp(-\delta |x|) \quad \text{for large} \ |x|.
\end{align*}

Then (2.4) has a positive solution in \(H^1(\mathbb{R}^N)\).

We can easily see (Q1)–(Q3) hold for \(K(e')\) by (K1)–(K3). Thus we can apply Theorem 5 with \(N = 1\) to our problem and we complete the proof of Theorem 1.

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**Added in proof**

After submitting this paper, we found that a related existence result was obtained in a paper of Bianchi and Egnell [BE].
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DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA 464-01, JAPAN

E-mail address: tanaka@math.nagoya-u.ac.jp

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