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A REMARK ON POSITIVE RADIAL SOLUTIONS OF THE ELLIPTIC EQUATION $\Delta u + K(|x|)u^{(n+2)/(n-2)} = 0$ IN $\mathbf{R}^n$

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ABSTRACT. We consider the following semilinear elliptic equation involving critical Sobolev exponents:

$$-\Delta u = K(|x|)u^{(n+2)/(n-2)} \quad \text{in } \mathbf{R}^n,$$
$$u(x) \to 0 \qquad \text{as } |x| \to \infty,$$

where $n \geq 3$, $K(r) \in C([0, \infty), \mathbf{R})$. We prove the existence of a positive radial solution with asymptotic behavior $C/|x|^{n-2}$ at $|x| = \infty$ under the conditions (i) $K(r) > 0$ for all $r > 0$, (ii) $K(0) = K(\infty)$, and (iii) there exist $C, \delta > 0$ such that $K(r) \geq K(0) - Cr^{\delta}$ for small $r > 0$ and $K(r) \geq K(0) - Cr^{-\delta}$ for large $r > 0$.

1. INTRODUCTION AND THE RESULT

In this paper, we consider the existence of positive radial solutions of the following semilinear elliptic equation involving critical Sobolev exponents:

$$-\Delta u = K(|x|)u^{(n+2)/(n-2)} \qquad \text{in } \mathbf{R}^n, \tag{1.1}$$
$$u(x) \to 0 \qquad \text{as } |x| \to \infty, \tag{1.2}$$

where $n \geq 3$, $K(r) \in C([0, \infty), \mathbf{R})$.

This problem arises in Riemannian geometry and the existence of positive radial solutions is deeply investigated by [N, DN, LL, LN, YY1, YY2]. In particular, [DN] showed the following striking phenomena: Let $\eta(r) \in C_0^\infty([0, \infty), \mathbf{R})$ be a function which is monotonically nonincreasing in r and satisfies $\eta(0) \in (0, 1)$, $\eta(r) \geq 0$. Then

(i) (1.1)–(1.2) possesses no positive radial solutions for $K(r) \equiv 1 - \eta(r)$;
(ii) (1.1)–(1.2) possesses infinitely many positive radial solutions for $K(r) \equiv 1+\eta(r)$ and all of them have the same asymptotic behavior $C/|x|^{(n-2)/2}$;

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(iii) (1.1)–(1.2) possesses infinitely many positive radial solutions for $K(r) \equiv 1$ and all of them have the same asymptotic behavior $C/|x|^{n-2}$.

This phenomena shows that if $K(0)$, $K(\infty) > 0$ and $K(0) \neq K(\infty)$, we cannot expect the existence of a positive radial solution with asymptotic behavior $C/|x|^{n-2}$ in general. In this paper we consider the existence of positive radial solutions with asymptotic behavior $C/|x|^{n-2}$ in the case

$$K(0) = K(\infty) > 0 \quad \text{and} \quad K(r) > 0 \text{ for all } r > 0. \tag{1.3}$$

Our main result is as follows:

Theorem 1. *Assume* $K(r) \in C([0, \infty), \mathbf{R})$ *satisfies*

(K1) $K(0) = K(\infty) \equiv \lim_{r\to\infty} K(r) > 0$;

(K2) $K(r) > 0$ *for all* $r > 0$;

(K3) *there are constants* δ, $C > 0$ *such that*

$$K(r) \geq K(0) - Cr^{+\delta} \quad \textit{for small } r,$$
$$K(r) \geq K(0) - Cr^{-\delta} \quad \textit{for large } r.$$

Then (1.1)–(1.2) *possesses at least one positive radial solution with asymptotic behavior* $C/|x|^{n-2}$.

We remark that [LL, YY1, YY2] also studied the situations (1.3). In particular, by the minimizing method Lin and Lin [LL] obtained the existence of a positive radial solution with asymptotic behavior $C/|x|^{n-2}$ in the conditions: $K(0) > 0$, $K(r) = K(0) + \beta r^\alpha + o(r^\alpha)$ for small $r > 0$, $\int_1^\infty K^-(r) r^{-n-1}\, dr < \infty$, and $K(\infty) \leq K(0)$, where $\alpha \in (0, n)$, $\beta > 0$. By shooting methods, [YY1, YY2] obtained the existence of a positive radial solution with asymptotic behavior $C/|x|^{n-2}$ under the conditions: $K(0) = K(\infty) > 0$ and $K(r) \geq K(0)$ for all r. We remark that [YY1, YY2] also obtained a sharp result on the structure of positive radial solutions.

To prove our theorem, we introduce a transformation

$$v(t) = e^{(n-2)t/2} u(e^t). \tag{1.4}$$

We see $u(x) = u(|x|)$ is a positive radial solution with asymptotic behavior $C/|x|^{n-2}$ if and only if $v(t)$ is a positive solution of

$$-v_{tt} + \left(\frac{n-2}{2}\right)^2 v = K(e^t) v^{(n+2)/(n-2)} \quad \text{in } \mathbf{R}, \qquad v \in H^1(\mathbf{R}). \tag{1.5}$$

We apply the recent remarkable existence result due to Bahri and Li [BYL] and Bahri and Lions [BPL] to (1.5). Finally we remark our transformation (1.4) was used in [DN] (in the proof of Theorem 5.33) to study the asymptotic behavior of solutions.

2. Proof of Theorem 1

We use variational methods to find a positive radial solution of (1.1)–(1.2). Let $D^{1,2}(\mathbf{R}^n)$ be the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} |\nabla u|^2\, dx\right)^{1/2},$$

and we define $E = \{u \in D^{1,2}(\mathbf{R}^n)\,;\, u(x) = u(|x|)$, i.e., u is a radial function$\}$ and

$$\|u\|_E = \left(\int_{\mathbf{R}^n} |\nabla u|^2\, dx\right)^{1/2}.$$

We consider the functional

$$I(u) = \int_{\mathbf{R}^n} \left[\frac{1}{2}|\nabla u|^2 - \frac{2n}{n-2}K(|x|)|u|^{2n/(n-2)}\right] dx \in C^2(E, \mathbf{R}).$$

It is known that critical points $u \in E$ with $u(x) \geq 0$ $(u(x) \not\equiv 0)$ are positive radial solutions of (1.1)–(1.2). As to the asymptotic behavior of $u(x)$ as $|x| \to \infty$, we consider the Kelvin transformation $\overline{u}(x)$ of $u(x)$, that is,

$$\overline{u}(x) = |x|^{-(n-2)} u\left(\frac{x}{|x|^2}\right). \tag{2.1}$$

We can verify $\overline{u}(x)$ is a critical point of

$$\overline{I}(u) = \int_{\mathbf{R}^n} [\tfrac{1}{2}|\nabla u|^2 - \tfrac{2n}{n-2}K(\tfrac{1}{|x|})|u|^{2n/(n-2)}]\, dx.$$

Thus $\overline{u}(x)$ is a positive solution of the corresponding equation, in particular, $\overline{u}(x) \in C(\mathbf{R}^n)$. Thus there exists a limit $\overline{u}(0) = \lim_{|x|\to 0} \overline{u}(x)$. That is, the limit $\lim_{|x|\to\infty} |x|^{n-2}u(x)$ exists. Therefore, critical points of $I(u)$ with $u(x) \geq 0$ $(u(x) \not\equiv 0)$ correspond to positive radial solutions with asymptotic behavior $C/|x|^{n-2}$.

For $u(x) = u(|x|) \in E$, we define a function $(Tu)(t)\colon \mathbf{R} \to \mathbf{R}$ by

$$(Tu)(t) = e^{(n-2)t/2}u(e^t).$$

Then we have

Proposition 2. *T is a bounded linear operator with a bounded inverse from E to $H^1(\mathbf{R})$. More precisely, T satisfies for all $u \in E$*

$$\|u\|_E^2 = \omega_{n-1}\left(\|(Tu)_t\|_{L^2(\mathbf{R})}^2 + \left(\frac{n-2}{2}\right)^2 \|Tu\|_{L^2(\mathbf{R})}^2\right). \tag{2.2}$$

Furthermore, T satisfies for all $u \in E$

$$\int_{\mathbf{R}^n} K(|x|)|u|^{2n/(n-2)}dx = \omega_{n-1}\int_{\mathbf{R}} K(e^t)|(Tu)(t)|^{2n/(n-2)}\, dt. \tag{2.3}$$

Here we used the notation: $\omega_{n-1} = \int_{S^{n-1}} d\sigma$, $S^{n-1} = \{x \in \mathbf{R}^n\,;\, |x| = 1\}$, and $\|u\|_{L^2(\mathbf{R})} = (\int_{\mathbf{R}} |u|^2\, dx)^{1/2}$.

Proof. Since $C_{0,r}^\infty(\mathbf{R}^n) = \{u \in C_0^\infty(\mathbf{R}^n)\,;\, u$ is a radial function, $0 \notin \operatorname{supp} u\}$ is dense in E, it suffices to verify (2.2) and (2.3) for $u \in C_{0,r}^\infty(\mathbf{R}^n)$. By the straightforward computation, we can get (2.2), (2.3) easily.

We can also easily verify the following properties of T.

Proposition 3. (i) $(Tu_\lambda)(t) = (Tu)(t+\log\lambda)$ *for all* $u \in E$, $\lambda > 0$, *and* t, *where* $u_\lambda(x) = \lambda^{(n-2)/2}u(\lambda x)$.

(ii) $(T\bar{u})(t) = (Tu)(-t)$ *for all* $u \in E$ *and* t, *where* $\bar{u}(x)$ *is the Kelvin transformation of* $u(x)$ *defined in* (2.1).

We introduce a functional $J(v)\colon H^1(\mathbf{R}) \to \mathbf{R}$ by

$$J(v) = \int_{\mathbf{R}} \left[\frac{1}{2} \left(|v_t|^2 + \left(\frac{n-2}{2} \right)^2 |v|^2 \right) - \frac{2n}{n-2} K(e^t)|v|^{2n/(n-2)} \right] dt.$$

By Proposition 2, we can see

$$I(u) = \omega_{n-1} J(Tu) \quad \text{for all } u \in E.$$

Thus we have

Proposition 4. *$u \in E$ is a critical point of $I(u)$ if and only if $(Tu)(t) \in H^1(\mathbf{R})$ is a critical point of $J(v)$, that is, $(Tu)(t)$ is a solution of* (1.5).

Recently there has been remarkable progress due to Bahri and Li [BYL] and Bahri and Lions [BPL] in the study of the existence of critical points of the functional:

$$F(u) = \int_{\mathbf{R}^N} \left[\frac{1}{2}(|\nabla u|^2 + m|u|^2) - \frac{1}{p+1} q(x)|u|^{p+1} \right] dx \in C^2(H^1(\mathbf{R}^N), \mathbf{R}),$$

where $m > 0$, $p \in (1, (N+2)/(N-2))$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1, 2$, and $q(x) \in L^\infty(\mathbf{R}^N)$. Critical points of $F(u)$ correspond to solutions of the following semilinear elliptic equations:

$$(2.4) \qquad -\Delta u + mu = q(x)|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N).$$

Their result is

Theorem 5 ([BYL, Theorem 3.1], cf. [BPL]). *Assume $q(x) \in L^\infty(\mathbf{R}^N)$ satisfies*
(Q1) $q(x) > 0$ *for all* x;
(Q2) $\lim_{|x|\to\infty} q(x) \equiv q_\infty > 0$ *exists*;
(Q3) *there exist constants* δ, $C > 0$ *such that*

$$q(x) \geq q_\infty - C\exp(-\delta|x|) \quad \textit{for large } |x|.$$

Then (2.4) *has a positive solution in* $H^1(\mathbf{R}^N)$.

We can easily see (Q1)–(Q3) hold for $K(e^t)$ by (K1)–(K3). Thus we can apply Theorem 5 with $N = 1$ to our problem and we complete the proof of Theorem 1.

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Added in proof

After submitting this paper, we found that a related existence result was obtained in a paper of Bianchi and Egnell [BE].

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