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BANACH SPACES WITH AN UNCONDITIONAL BASIS THAT ARE ISOMORPHIC TO A NONATOMIC BANACH LATTICE

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ABSTRACT. It is proved that if X is a Banach space with an unconditional basis, then X is isomorphic to a nonatomic Banach lattice which is not order continuous if and only if X contains a subspace isomorphic to c_0.

Finite Borel measures on separable, complete metric spaces come in three types: Legesgue measure on $[0, 1]$ (nonatomic measures), counting measure on the integers (atomic measures), and a combination of the first two [R]. It is the same with Banach lattices. An atom in a Banach lattice is an element $a > 0$ such that for all $b > 0$ with $a > b$, b is a scalar multiple of a. Purely atomic (discrete) Banach lattices are those for which every positive element dominates some atom. Purely nonatomic (continuous) Banach lattices are those that have no atoms. So, Banach lattices, like nice measures, come in three types: the purely atomic ones, the purely nonatomic ones, and those that are a sum of the two. The question considered in this paper is: when are purely atomic Banach lattices isomorphic to purely nonatomic ones? The scope of this paper is to consider only purely atomic lattices with an unconditional basis giving the lattice structure. That is, the atoms are the basis elements and positive multiples thereof.

Recently, Kalton and Wojtaszczyk [KW] characterized those Banach spaces with an unconditional basis which are isomorphic to order-continuous, (purely) nonatomic Banach lattices. An order-continuous Banach lattice is one where every downwardly directed net which converges in order to 0 also converges in norm to 0. Additionally, it is known that every lattice structure on a Banach lattice not containing c_0 is order-continuous [LT2, pp. 6–8]. [KW] showed that a Banach lattice X with unconditional basis is isomorphic to a nonatomic order-continuous Banach lattice if and only if $X \approx X(l_2)$. However, their paper does not address the question of when a space is isomorphic to a nonatomic Banach lattice which is not order-continuous. Since E. Lacey and P. Wojtaszczyk [LW]

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showed that c_0 is isomorphic to a nonatomic, not order-continuous Banach lattice; one might guess that X has the same property if and only if $X \approx X(c_0)$. While it is certainly true that $X(c_0)$ is isomorphic to a nonatomic, not order-continuous Banach lattice, the necessity is false. As an example, consider $X = l_2 \oplus c_0$; this X is isomorphic to a nonatomic, not order-continuous, Banach lattice, since both l_2 and c_0 are, but $X \not\approx X(c_0)$.

Before the paper [KW], it was known [AW] that l_1 is not isomorphic to a nonatomic Banach lattice while l_2 and (by [LW]) c_0 are. So it is interesting to consider the cases of

$$l_1 \oplus l_2 \quad \text{and} \quad l_1 \oplus c_0 .$$

The results of [KW] imply that $l_1 \oplus l_2$ is not isomorphic to a nonatomic Banach lattice, since it does not contain c_0 and hence every lattice structure on it is order-continuous. A surprising consequence of Theorem 1 (below) is that $l_1 \oplus c_0$ is isomorphic to a nonatomic, not order-continuous Banach lattice. So there is something special in the structure of c_0 that l_2 does not have. This paper characterizes those Banach spaces with an unconditional basis which are isomorphic to nonatomic Banach lattices which are not order-continuous.

Theorem 1. *Let X be a Banach space with an unconditional basis. Then X is isomorphic to a nonatomic Banach lattice which is not order-continuous if and only if X contains a subspace isomorphic to c_0.*

The necessity is clear from the fact that Banach lattices not containing c_0 are order-continuous. The proof of sufficiency consist of the construction of a nonatomic, not order-continuous Banach lattice Y and then several lemmas that show that $X \approx Y$.

Construction

The space Y is based on the construction by E. Lacey and P. Wojtaszczyk [LW] of a nonatomic Banach lattice isomorphic to c_0. In [LW], the desired space is a certain collection of continuous functions on a certain compact space T, under the sup norm. Our construction will use the same space T but modify the collection of continuous functions; our norm depends on X and is the sup norm when $X = c_0$.

T is a space of closed intervals in the first quadrant of the plane. These intervals are built in levels, and each of the intervals originate at the point $(0, 0)$. To construct the intervals, first pick a number $0 < r < 1$. Let I_0 be the unit interval along the y-axis. This is the 0th level. For the first level, pick a sequence of intervals (originating at $(0, 0)$) of length r whose slopes increase to $+\infty$. Call these intervals $I_{0,n}$, where n increases with the slopes of the intervals. On the second level, for each n, pick a sequence of intervals of length r^2 whose slopes converge to the slope of $I_{0,n}$, and which lie between $I_{0,n}$ and $I_{0,n-1}$ whenever $n > 1$. Continue in this manner, so that at the nth level we have sequences of intervals of length r^n. The compact set T is the union of all of these intervals. (See Figure 1.)

Now let S_n be the part of the circle $x^2 + y^2 = (r^n)^2$ that lies in the first quadrant. Let $\{w_i\}_{i=1}^{\infty} = (S_1 \cap T)\backslash\{(0, r)\}$. Let $\{e_i\}_{i=1}^{\infty}$ be the unconditional

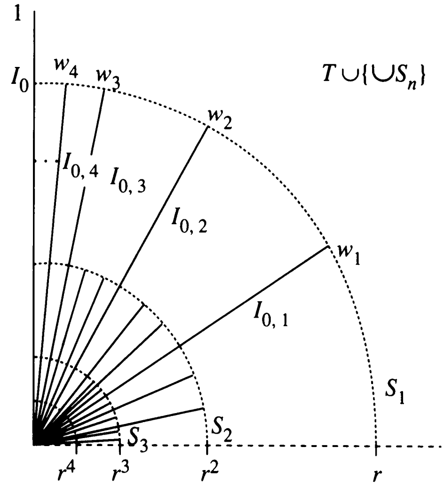

FIGURE 1

basis for X. We define the space Y by

$$Y = \Big\{ f \in C(T) : f(0, \alpha) = 0 \text{ for } 0 \leq \alpha \leq 1, \\ f \text{ is affine on intervals, } \sum_i f(w_i)e_i \in X \Big\} .$$

Let $\|\cdot\|_Y$ be defined for $f \in Y$ by

$$\|f\|_Y = \max\left\{ \left\| \sum_i f(w_i)e_i \right\|_X , \sup_{t\in T} |f(t)| \right\} ,$$

and let the order on Y be the natural order; that is, $f \leq g$ if and only if $f(t) \leq g(t)$ for all $t \in T$. The following is a fact about spaces with unconditional bases that we will use often.

Lemma 2. *For each $w_k \in S_1 \cap T$ and for each $f \in Y$, $|f(w_k)| \leq \| \sum_i f(w_i)e_i\|_X$.*

Proof. See [LT1, Proposition 1.c.7]. □

It is easy to see that Y is a Banach lattice when endowed with the norm and order given above and when you consider that X is a Banach lattice with the order given by its unconditional basis $\{e_i\}_{i=1}^{\infty}$. To see that Y is nonatomic, consider a function $0 < f \in Y$. Recalling that f is continuous on T, f is affine on intervals, and every interval in T has an initial segment that is the limit of other, shorter intervals, we see that f cannot be supported entirely on only one interval. Take intervals I_1 and I_2 in the support of f. Clearly, it is possible to choose $0 < g \in Y$ so that $g < f$, $g = 1/2f$ on I_1, and $g = 1/3f$ on I_2. This function g is less than f but not a multiple of f, hence f is not an atom.

Now for each $s \in S_n$, either $s \in T \cap S_n$, or there are closest s' and s'' in $T \cap S_n$ such that $\arg s' < \arg s < \arg s''$. Let J_s be the interval connecting

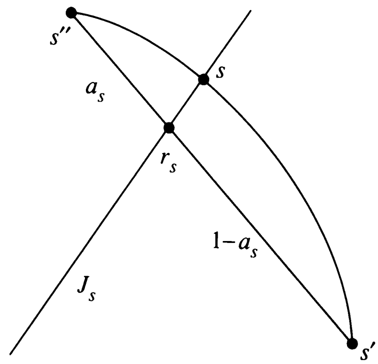

FIGURE 2

s to the point $(0, 0)$, and let r_s be the intersection of J_s with the chord connecting s' and s''. Then there is a number a_s with $0 < a_s < 1$ such that $r_s = a_s s' + (1 - a_s)s''$. (See Figure 2.)

For $f \in Y$ and $s \in S_n$, define

$$\Phi_n(f)(s) = a_s f(s') + (1 - a_s) f(s'')$$

with a_s, s', s'' defined as above. Then $\Phi_n \colon Y \to C(S_n)$ is an extension operator; that is, Φ_n takes $f|_{S_n \cap T}$ and extends it to a continuous function on S_n without increasing $\sup|f|$. Define $Q_n \colon Y \to Y$ by

$$Q_n(f)(t) = (|t|/r^n)\Phi_n(f)(r^n t/|t|).$$

Q_n is a projection on Y. $Q_n(Y)$ is the collection of functions on Y which are completely determined by their values on $T \cap S_n$.

Lemma 3. $\|Q_n\| = 1$.

Proof. Let $f \in Y$ with $\|f\| = 1$. Then $\|Q_n f\|_Y = \max\{\|\sum_i Q_n f(w_i) e_i\|_X, \max_t |Q_n f(t)|\}$. If $t \in T$ with $|t| < r^n$, then

$$|Q_n f(t)| = \frac{|t|}{r^n}\left|\Phi_n(f)\left(\frac{r^n t}{|t|}\right)\right| \le 1$$

because $|t|/r^n < 1$ and $|f(t)| \le 1$ for all t. If $1 \ge |t| \ge r^n$, then $r^n t/|t| \in S_n \cap T$ so that $\Phi_n(f)(t) = f(t)$. Hence,

$$|Q_n f(t)| = \frac{|t|}{r^n}\left|f\left(\frac{r^n t}{|t|}\right)\right| = |f(t)| \le 1.$$

Finally,

$$\left\|\sum_i \frac{r}{r^n}\Phi_n f\left(\frac{r^n w_i}{r}\right) e_i\right\| = \left\|\sum_i \frac{r}{r^n} f\left(\frac{r^n w_i}{r}\right) e_i\right\| = \left\|\sum_i f(w_i) e_i\right\| \le 1. \quad \square$$

Lemma 4. $\lim_{n\to\infty} \|Q_n f - f\| = 0$ *for any* $f \in Y$.

Proof. The previous two lemmas and the definition of Q_n show that if $t \in T$ and $|t| \ge r^n$, then $Q_n f(t) = f(t)$ and if $|t| < r^n$, then $|Q_n f(t)| \le \max_{s \in S_n \cap T} |f(s)|$. Let $f \in Y$ and let $\varepsilon > 0$. Since f is continuous at $(0, 0)$, we can choose N so large that if $|t| \leq r^N$, then $|f(t)| < \varepsilon$. Now $Q_n f(t) = f(t)$ whenever $|t| \geq r^n$. So if $n > N$, then

$$\left\| \sum_i (Q_n f(w_i) - f(w_i)) e_i \right\| = 0$$

and

$$\max_t |Q_n f(t) - f(t)| \leq \max_{t \in S_n \cap T} |f(t)| + \sup_{|t| < r^N} |f(t)| \leq 2\varepsilon,$$

so that $\|Q_n f - f\| < 2\varepsilon$. □

It is easy to see that $Q_n Q_m = Q_{\min(n,m)}$. If we let $Q_0 = 0$, then the above facts show that $Y = \sum_n (Q_n - Q_{n-1})(Y)$. If we let $Y_n = (Q_n - Q_{n-1})(Y)$, then it is easy to see that Y_n is the set of all $f \in Y$ such that f is completely determined by its values on S_n and $f = 0$ on intervals of length greater than r^n. Hence, if we give Y_1 the natural norm inherited from X and give Y_n the sup norm for each n, then $Y_1 = Q_1(Y) \approx X$, $Y_2 = (\sum \bigoplus c_0)_{c_0} = c_0$, and $Y_n = (\sum \bigoplus Y_{n-1})_{c_0} = (\sum \bigoplus c_0)_{c_0}$ for $n > 2$.

Next, define a mapping T taking Y to $(\sum_n \bigoplus Y_n)_{c_0}$ by $T(f) = (Q_1(f), (Q_2 - Q_1)f, (Q_3 - Q_2)f, \ldots)$. T is a bounded linear operator since

$$\begin{aligned} \|Tf\| &= \max\{\|Q_1 f\|, \|(Q_2 - Q_1)f\|, \|(Q_3 - Q_2)f\|, \ldots\} \\ &\leq \max\{\|f\|, 2\|f\|, 2\|f\|, \ldots\} \leq 2\|f\|. \end{aligned}$$

Lemma 5. *T^{-1} is bounded.*

Proof. $T^{-1}(f_1, f_2, f_3, \ldots) = \sum_n f_n$. The proof that $\sum_n f_n$ is defined is contained in the proof that T^{-1} is bounded. Now

$$\|T^{-1}(f_1, f_2, f_3, \ldots)\| = \left\| \sum_n f_n \right\| = \max \left\{ \left\| \sum_i f_1(w_i) e_i \right\|_X, \max_t \left| \sum_i f_i(t) \right| \right\}.$$

Clearly, $\|\sum_i f_1(w_i) e_i\|_X \leq \|(f_1, f_2, f_3, \ldots)\|$. For the rest, let $t \in T$ be fixed. Then there is a largest k such that t belongs to an interval of length r^k. So, for this t

$$\left| \sum_i f_i(t) \right| \leq \sum_i |f_i(t)| \leq \sum_{i=1}^k |f_i(t)|$$

since $f_i \equiv 0$ on any interval of length greater than r^i. Now for each i, since $|t| \leq r^k \leq r^i$ and f is affine on intervals,

$$|f_i(t)| = \frac{|t|}{r^i} \left| f_i \left(\frac{t}{|t|} r^i \right) \right| \leq r^{k-i} \sup_{s \in S_i \cap T} |f_i(s)|.$$

Thus, if $i > 1$, $|f_i(t)| \leq r^{k-i} \|f_i\|$ and if $i = 1$, we may apply Lemma 2 to get $|f_i(t)| \leq r^{k-i} \|f_i\|$. Hence,

$$\begin{aligned} \sum_{i=1}^k |f_i(t)| &\leq \sum_{i=1}^k \|f_i\| r^{k-i} \leq \|(f_1, f_2, f_3, \ldots)\| \sum_{j=0}^{\infty} r^j \\ &\leq \|(f_1, f_2, f_3, \ldots)\| \frac{1}{1-r}, \end{aligned}$$

so that T^{-1} is bounded. □

This shows that $Y \approx (\sum_n \bigoplus Y_n)_{c_0} \approx X \oplus \sum_n c_0 \approx X \oplus c_0$. The fact that Y is not order-continuous is easy to see from its construction (Y contains lattice copies of c) or from the fact that the only order-continuous order on c_0 is the canonical one [AW], and $c_0 = (\sum_{n=2}^{\infty} \bigoplus Y_n)_{c_0}$ does not have the canonical order. The following well-known lemma will finish the proof of the theorem.

Lemma 6. $X \approx X \oplus c_0$.

Proof. Since X contains c_0, then X contains c_0 complementably (see [LT1, 2.f.5]). Thus, $X \approx Y \oplus c_0$ for some $Y \subset X$. However, $c_0 \approx c_0 \oplus c_0$ so that $X \approx Y \oplus (c_0 \oplus c_0) \approx (Y \oplus c_0) \oplus c_0 \approx X \oplus c_0$. □

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