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CONVEX BODIES AND CONCAVE FUNCTIONS

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ABSTRACT. We find properties that a class $\mathfrak{C}$ of closed bounded convex subsets of a Banach space E and a mapping $p: \mathfrak{C} \to \mathbb{R}_+$ should satisfy in order to obtain the following result:

Theorem. *Let $\mathfrak{C}$ and $p: \mathfrak{C} \to \mathbb{R}_+$ satisfy these properties, and let K be a closed convex subset of $[0, 1] \times E$ such that for every $t \in [0, 1]$ the set $K(t) = \{z \in E; (t, z) \in K\}$ is an element of $\mathfrak{C}$. Suppose that a concave continuous function $f: [0, 1] \to \mathbb{R}$ is given such that*

$$0 \leq f(t) \leq p(K(t)), \quad \text{for every } t \in [0, 1].$$

Then there exists a closed convex subset L of $[0, 1] \times E$ such that $L \subset K$,

$$L(t) = \{z \in E; (t, z) \in L\} \in \mathfrak{C} \quad \text{and} \quad f(t) = p(L(t)) \text{ for every } t \in [0, 1].$$

Some examples and applications are given to the study of Steiner symmetrization and of the Riesz decomposition property for concave continuous functions.

INTRODUCTION

The aim of this paper was originally to study the following problem: Let A be a convex body in $\mathbb{R}^d$ and $\widetilde{A}$ be its Steiner symmetrical with respect to some fixed hyperplane H. If C is a convex body such that $C \subset \widetilde{A}$ and C is symmetric with respect to H, does there exist a convex body $B \subset A$ such that the Steiner symmetral $\widetilde{B}$ of B with respect to H is equal to C? Using the general result (Theorem 2) mentioned in the abstract, we prove that the answer is yes if $d = 2$ and generally no if $d \geq 3$. We study also the following problem (Corollary 3): given a direction $u \in S_{d-1}$ and a convex body A in $\mathbb{R}^d$ such that all its sections by hyperplanes orthogonal to u are homothetical to some fixed convex body D, how does one characterize the convex bodies $B \subset A$ with the same property? We show (Corollary 4) that, given any convex body A in $\mathbb{R}^d$, there exists a convex body $B \subset A$ whose sections by hyperplanes orthogonal to some fixed direction u have prescribed mean width.

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Definition. Let E be a normed space, and let $\mathfrak{C}$ be a class of closed bounded convex subsets of E such that:

$\varnothing \notin \mathfrak{C}$ and there exists $C \in \mathfrak{C}$ with nonempty interior.

$\lambda C + \mu D \in \mathfrak{C}$ for every C, $D \in \mathfrak{C}$ and λ, $\mu \geq 0$.

$x + C \in \mathfrak{C}$ for every $x \in E$ and $C \in \mathfrak{C}$.

For every family $\mathscr{F}$ of elements of $\mathfrak{C}$, totally ordered by inclusion,

$$\bigcap_{C \in \mathscr{F}} C \in \mathfrak{C}.$$

Let $p \colon \mathfrak{C} \to \mathbb{R}_+$ be such that:

p is increasing: if C, $D \in \mathfrak{C}$, $C \subset D$, then $p(C) \leq p(D)$.

$p(\lambda C + \mu D) = \lambda p(C) + \mu p(D)$ for every C, $D \in \mathfrak{C}$ and λ, $\mu \geq 0$.

$p(x + C) = p(C)$ for every $x \in E$ and $C \in \mathfrak{C}$.

For every family $\mathscr{F}$ of elements of $\mathfrak{C}$, totally ordered by inclusion,

$$p\left(\bigcap_{C \in \mathscr{F}} C\right) = \inf_{C \in \mathscr{F}} p(C).$$

If $(E, \mathfrak{C}, p)$ satisfies all these properties, we shall say that $(E, \mathfrak{C}, p)$ is *admissible.*

Lemma 1. *Let h and f be two concave continuous functions on $[0, 1]$ such that $0 \leq f \leq h$ and $f \neq 0$, h. Then there exists a concave continuous function g on $[0, 1]$ satisfying $f \leq g \leq h$, $g \neq f$, h, and at least one of the following properties*:

(1) *For some a, $b \in [0, 1]$, $a < b$, g is affine on $[a, b]$ and coincides with h on $[0, a] \cup [b, 1]$.*

(2) *g is affine on $[0, 1]$, and $g = \rho h$ for some $\rho \in]0, 1[$.*

(3) *$g(0) = \rho h(0)$ for some $\rho \in]0, 1[$, and for some $a \in]0, 1[$, g is affine on $[0, a]$ and coincides with h on $[a, 1]$.*

(4) *$g(1) = \rho h(1)$ for some $\rho \in]0, 1[$, and for some $b \in [0, 1[$, g is affine on $[b, 1]$ and coincides with h on $[0, b]$.*

Proof. Let $]\alpha_n, \beta_n[$, $n \in \mathbb{N}$, be the interiors in $\mathbb{R}$ of the connected components of the set $\{f < h\} = \{t \in [0, 1];\ f(t) < h(t)\}$; then we have $\{f < h\} \cap]0, 1[= \bigcup_{n \in \mathbb{N}}]\alpha_n, \beta_n[$; the proof can be reduced to two cases:

I. For some $n \in \mathbb{N}$, there exists $t_0 \in]\alpha_n, \beta_n[$ such that, for every neighborhood V of t_0, h is not affine on V. Then by the continuity of f and h, for some $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset]\alpha_n, \beta_n[$, we have

$$\theta h(t_0 - \varepsilon) + (1 - \theta) h(t_0 + \varepsilon) > \max_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} f(t)$$

for every $\theta \in [0, 1]$. Then we define $[a, b] = [t_0 - \varepsilon, t_0 + \varepsilon]$ and we take the function g as defined in (1). Since h is concave but not affine on $[t_0 - \varepsilon, t_0 + \varepsilon]$, we have $g < h$.

II. For every $n \in \mathbb{N}$ and every $t \in]\alpha_n, \beta_n[$, h is affine on some neighborhood of t. Then by compactness and connectivity, using the continuity of h, we get that h is affine on each of the intervals $[\alpha_n, \beta_n]$. If $0 < \alpha_n < \beta_n < 1$, we have $f(\alpha_n) = h(\alpha_n)$ and $f(\beta_n) = h(\beta_n)$; it follows then from the concavity of f that $f = h$ on $]\alpha_n, \beta_n[$, which is of course absurd from the definition of $]\alpha_n, \beta_n[$. Thus one of the following properties holds:

(a) $\{f < h\} = [0, 1]$ and h is affine on $[0, 1]$.

(b) For some $0 < \alpha \leq 1$, $h(\alpha) = f(\alpha)$, $[0, \alpha[\subset \{f < h\}$, and h is affine on $[0, \alpha]$.

(c) For some $0 \leq \beta < 1$, $h(\beta) = f(\beta)$, $]\beta, 1] \subset \{f < h\}$, and h is affine on $[\beta, 1]$.

By continuity, case (a) reduces to conclusion (2).

In case (b), by the continuity of f, for some $0 < \varepsilon < \alpha$ and $0 < \rho < 1$, we have

$$\max_{t \in [0, \varepsilon]} f(t) \leq \rho \min_{t \in [0, \varepsilon]} h(t).$$

Taking $a = \varepsilon$, we define g satisfying property (3). Case (c) is completely analogous to case (b) and yields property (4). □

Theorem 2. *Let $(E, \mathfrak{C}, p)$ be admissible, and let K be a closed convex subset of $[0, 1] \times E$ such that for every $t \in [0, 1]$ the set $K(t) = \{z \in E ; (t, z) \in K\}$ is an element of $\mathfrak{C}$. Suppose that a concave continuous function $f: [0, 1] \to \mathbb{R}$ is given such that*

$$0 \leq f(t) \leq p(K(t)), \quad \textit{for every } t \in [0, 1].$$

Then there exists a closed convex subset L of $[0, 1] \times E$ such that $L \subset K$,

$$L(t) = \{z \in E ; (t, z) \in L\} \in \mathfrak{C} \quad \textit{and} \quad f(t) = p(L(t)) \textit{ for every } t \in [0, 1].$$

Proof. Let $\mathfrak{D}$ be the set of all closed convex subsets M of K such that for every $t \in [0, 1]$, $M(t) \in \mathfrak{C}$ and $p(M(t)) \geq f(t)$. Then it is easy to see that, ordered by inclusion, $\mathfrak{D}$ is inductive. By Zorn's lemma, $\mathfrak{D}$ thus has a minimal element, say N; define $h: [0, 1] \to \mathbb{R}_+$ by $h(t) = p(N(t))$. It is clear that f and h are concave and continuous on $[0, 1]$ and satisfy $h \geq f$; we shall show that $h = f$. Suppose $f < h$; then given the function g obtained by Lemma 1, we shall construct $N' \in \mathfrak{D}$, $N' \subset N$, such that $g(t) = p(N'(t))$ for every $t \in [0, 1]$. But, since $f < g < h$, this contradicts the minimality of N. Let us consider the four cases of Lemma 1:

(1) Let $N'(t) = N(t)$ if $t \in [0, a] \cup [b, 1]$ and $N'(\lambda a + (1 - \lambda) b) = \lambda N(a) + (1 - \lambda) N(b)$, for $0 \leq \lambda \leq 1$. Then define $N' = \{(t, z) ; t \in [0, 1], z \in N'(t)\}$. It is clear that $N' \in \mathfrak{D}$, $N' \subset N$, and $p(N'(t)) = g(t)$ for every $t \in [0, 1]$.

(2) Let $x_0 \in N(0)$ and $x_1 \in N(1)$, and define for $t \in [0, 1]$, $N'(t) = (1 - t)(x_0 + \rho(N(0) - x_0)) + t(x_1 + \rho(N(1) - x_1))$.

(3) Let $x \in N(0)$; define $N'(t) = (1 - \frac{t}{a})(x + \rho(N(0) - x)) + \frac{t}{a} N(a)$ for $t \in [0, a]$ and $N'(t) = N(t)$ for $t \in [a, 1]$.

(4) is analogous to (3). □

Remark. If we had a concave continuous function f defined on $[a, b]$, $0 \leq a \leq b \leq 1$, satisfying $0 \leq f(t) \leq p(K(t))$ for every $t \in [a, b]$, the conclusion of Theorem 2 would still hold, with $L(t) = \varnothing$ for $t \notin [a, b]$.

Examples. We give some examples of admissible $(E, \mathfrak{C}, p)$. We refer to [B-Z] for all the undefined terminology about convex sets.

(1) Let E be a Banach space, and let D be a bounded closed convex subset of E with 0 in its interior; let $\mathfrak{C}_D$ be the family of all subsets of E of the form $x + \mu D$, $x \in E$, $\mu \geq 0$; and let $p_D: \mathfrak{C}_D \to \mathbb{R}_+$ be defined by $p_D(x + \mu D) = \mu$; observe that for λ, $\mu \geq 0$, one has $x + \mu D \subset y + \lambda D$ if and only if $\mu \leq \lambda$ and $x - y \in (\lambda - \mu) D$; it is easy to verify, using the completeness of E, that $(E, \mathfrak{C}_D, p_D)$ is admissible.

(2) Let $\mathscr{K}^d$ be the class of all compact convex subsets of $E = \mathbb{R}^d$. If $p: \mathscr{K}^d \to \mathbb{R}_+$ satisfies the first three properties of admissible mappings, then it follows from Hahm-Banach theorem and the Riesz representation theorem that

$$p(K) = \int_{S^{d-1}} H_K(u)\, d\nu(u) \quad \text{for } K \in \mathscr{K}^d$$

for some positive measure ν on the sphere S^{d-1} such that $\int_{S^{d-1}} u\, d\nu(u) = 0$ (H_K denotes the support function of K: $H_K(u) = \max\{\langle x, u\rangle ; x \in K\}$, where $\langle x, u\rangle$ is the scalar product of x, $u \in \mathbb{R}^d$). In particular such a p satisfies automatically the fourth property of admissible mappings (see [F]). For instance, if K_j, $2 \le j \le d$, are convex bodies in $\mathbb{R}^d$, one can consider as $p(K)$, the mixed volume $V(K, K_2, \dots, K_d)$ of K together with $K_2, \dots, K_p$ or

$$V(K, B, \dots, B) = h_m(K) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} H_K(u)\, d\sigma(u),$$

where B is the Euclidean ball and σ is the rotation invariant measure on S^{d-1} with total mass ω_{d-1} (this is the mean width of K, and it coincides with $1/2\pi$ times the perimeter of K if $d = 2$).

As applications of Theorem 2 to these examples, we get the following corollaries.

Corollary 3. *Let A be a body in $\mathbb{R}^d$, such that for some axis of direction $u \in S^{d-1}$, the sections $A(t) = \{z \in A ; \langle z, u\rangle = t\}$ are all homothetic (whenever they are nonempty) to some fixed convex body $D \subset \{z ; \langle z, u\rangle = 0\}$. Let f be a nonnegative continuous function on $[a, b] \subset [0, 1]$ such that $f^{1/(d-1)}$ is concave and*

$$f(t) \le \operatorname{vol}(A(t)) \quad \textit{for every } t \in [a, b],$$

where $\operatorname{vol}(\cdot)$ *denotes the volume in hyperplanes orthogonal to u. Then there exists a body B such that $B \subset A$, and for $t \in [a, b]$, the sections $B(t) = \{z \in B ; \langle z, u\rangle = t\}$ are homothetic to D and satisfy $f(t) = \operatorname{vol}(B(t))$.*

Proof. With the notation of Example (1), apply Theorem 2 to $E = \mathbb{R}^{d-1}$, $\mathfrak{C} = \mathfrak{C}_D$, and $p = p_D$. □

Remark. If the sections $A(t)$ of A are not all homothetic, then Corollary 3 is no longer true, even if the problem is only to find a convex body $B \subset A$ such that $f(t) = \operatorname{vol}(B(t))$ for every $t \in [a, b]$:

Embed $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$ by $X = (X, 0)$, and suppose that $A = \operatorname{conv}(A_0, A_1)$, where A_0 and A_1 are two *nonhomothetic* convex bodies of R^{d-1} and conv(,) means convex hull. Then for $t \in [0, 1]$, $(A(t), t) = ((1-t)A_0 + tA_1, t)$, and if

$$f(t) = ((1-t)\operatorname{vol}(A_0)^{1/(d-1)} + t \operatorname{vol}(A_1)^{1/(d-1)})^{d-1},$$

then $f^{1/(d-1)}$ is concave (in fact, even affine) on $[0, 1]$, $f(0) = \operatorname{vol}(A_0)$ and $f(1) = \operatorname{vol}(A_1)$. By the Brunn-Minkowski theorem and its equality case, since A_0 and A_1 are not homothetic,

$$0 < f(t) < \operatorname{vol}(A(t)) \quad \text{for every } t \in]0, 1[.$$

But if a convex body B in $\mathbb{R}^d$ satisfies $B \subset A$ and $\operatorname{vol}(B(t)) \ge f(t)$ for every $t \in [0, 1]$, then clearly $B(0) = A_0$ and $B(1) = A_1$, so that $A = \operatorname{conv}(A_0, A_1) \subset B \subset A$. It follows that $\operatorname{vol}(B(t)) > f(t)$ for every $t \in]0, 1[$.

Corollary 4. *Let A be a convex body in $\mathbb{R}^d$, and let $u \in S^{d-1}$. Let f be a nonnegative continuous concave function on $[a, b]$ such that for every $t \in [a, b]$ we have*

$$f(t) \leq h_m(\{z \in A; \langle z, u\rangle = t\}).$$

Then there exists a convex body $B \subset A$ such that $f(t) = h_m(\{z \in B, \langle z, u\rangle = t\})$ for every $t \in [a, b]$ and $\{z \in B; \langle z, u\rangle = t\} = \varnothing$ if $t \notin [a, b]$.

Proof. With the notation of Example (2), apply Theorem 2 to $E = \mathbb{R}^{d-1}$, $\mathfrak{C} = \mathscr{K}^{d-1}$, and $p = h_m$ (mean width). □

Remark. The preceding corollary, applied to the case $d = 3$, allows us to find bodies inside A, with given (by f) perimeters of all the plane sections orthogonal to some fixed direction.

Let us apply Corollary 3 with $d = 2$: if A is a convex body in $\mathbb{R}^2$, then for some interval $[a, b]$ of $\mathbb{R}$ and some concave continuous functions g_1 and g_2 on $[a, b]$ satisfying $g_1 + g_2 \geq 0$, we can write

$$A = \{(x, y); x \in [a, b], -g_2(x) \leq y \leq g_1(x)\}.$$

Now if f is any concave continuous function on $[a, b]$ such that

$$0 \leq f(x) \leq g_1(x) + g_2(x) \quad \text{for every } x \in [a, b],$$

Corollary 3 says that there is a convex body $B \subset A$ such that for every $x \in [a, b]$,

$$f(x) = \max\{y; (x, y) \in B\} - \min\{y; (x, y) \in B\}.$$

This means also that there exists concave continuous functions h_1 and h_2 on $[a, b]$ such that $h_1 \leq g_1$, $h_2 \leq g_2$, and $f = h_1 + h_2$. This is the so-called Riesz decomposition property for concave continuous functions on $[a, b]$, with the pointwise order (clearly the hypothesis $g_1 + g_2 \geq 0$ is here irrelevant). The proof of Theorem 2 is inspired from this classical result in potential theory (see [M-S] and also [A]). Observe that the Riesz decomposition property can also be proved by using the unique decomposition of the extreme points of the order-segment $[0, g_1 + g_2] = \{f; f$ concave continuous, $0 \leq f \leq g_1 + g_2\}$ and then the Krein-Milmann theorem.

It should be noticed that this property is no longer true for functions of more than one variable:

Proposition 5. *Let K be a convex body in $\mathbb{R}^d$, $d \geq 1$. Suppose that the cone $V(K)$ of all the continuous concave functions on K satisfies the Riesz decomposition property, that is: for any f, g_1, $g_2 \in V(K)$ such that $f \leq g_1 + g_2$, there exists h_1, $h_2 \in V(K)$ such that $h_1 \leq g_1$, $h_2 \leq g_2$, and $f = h_1 + h_2$. Then $d = 1$ (and K is a segment).*

Proof. We shall first prove that if K is a convex body in $\mathbb{R}^2$, $V(K)$ does not satisfy this property; then we extend this result to any value of $d \geq 3$ (for $d = 1$, we refer to the above comments).

(1) Let P be a convex body in $\mathbb{R}^2$, and select two points X_1 and X_2 in P such that if Y_1Y_2 is the chord of P passing through X_1 and X_2, then $d(Y_1, X_1) = d(X_2, Y_2)$ (where $d(\;,\;)$ denotes the Euclidean distance); moreover, we can suppose that in the case when P is a quadrangle Y_1Y_2 is not a diagonal of P.

Embed $\mathbb{R}^2$ into $\mathbb{R}^3$ by $X = (X, 0)$. We define a convex body A of $\mathbb{R}^3$ by

$$A = \operatorname{conv}(P, (X_1, 1), (X_2, -1))$$

and two functions $g_1, g_2 \in V(P)$ by the identity

$$A = \{(X, t); X \in P, -g_2(X) \le t \le g_1(X)\}.$$

Let $a = g_1(X_2) = g_2(X_1) > 0$ and $C = \operatorname{conv}(P, (X_1, 1+a), (X_2, 1+a))$, and for $X \in P$ define $f(X) = \max\{t; (X, t) \in C\}$; it is clear that $f \in V(P)$, $f = 0$ on ∂P (the boundary of P), $f(X_1) = f(X_2) = 1 + a$, and $0 \le f \le g_1 + g_2$. Thus, if f could be written as $f = h_1 + h_2$, with $h_i \in V(P)$ and $h_i \le g_i$ for $i = 1, 2$, we would have $h_i = g_i$ on ∂P and in X_1, X_2, for $i = 1, 2$, and thus $h_i = g_i$ on all P. Under our assumptions, this is impossible for the following reasons.

Since X_1 and X_2 are interior points of P, there exist two points $Z_1, Z_2 \in \partial P$, lying respectively in the two open half-planes separated by the line X_1X_2, such that the line through Z_i parallel to X_1X_2 supports P at Z_i, $i = 1, 2$. It follows that the hyperplane of $\mathbb{R}^3$ passing through Z_i, $(X_1, 1)$, and $(X_2, 1)$ supports C in Z_i, $i = 1, 2$. Thus f is affine on each of the triangles $Z_iX_1X_2$, $i = 1, 2$. The equality $f = h_1 + h_2 = g_1 + g_2$ implies then that g_1 and g_2 are affine on each of these triangles; but then, since the graphs of g_1 and g_2 are cones with respective vertices $(X_1, 1)$ and $(X_2, 1)$ and basis P, the segments Y_iZ_j $(i, j = 1, 2)$ would be in ∂P. This implies that P is the quadrangle $Y_1X_1Y_2X_2$, which has X_1X_2 as a diagonal. We get a contradiction with the hypothesis.

(2) If K is a convex body in $\mathbb{R}^d$, select a two-dimensional affine subspace E of $\mathbb{R}^d$, passing through the interior of K, and define $P = E \cap K$. Let

$$A' = \operatorname{conv}(K, (X_1, 1), (X_2, -1)),$$

and

$$C' = \operatorname{conv}(K, (X_1, 1+a), (X_2, 1+a)),$$

where X_1 and X_2 are chosen in $P \subset E$, like they were in the preceding paragraph. Then it is clear that $A' \cap (E \times \mathbb{R}) = \operatorname{conv}(P, (X_1, 1), (X_2, -1))$ and $C' \cap (E \times \mathbb{R}) = \operatorname{conv}(P, (X_1, 1+a), (X_1, 1+a))$. If we define f', g_1', g_2' with respect to C' and A', we get $f'|_E = f$ and $g_i'|_E = g_i$, $i = 1, 2$, and we can apply the preceding result. □

Let A be a convex body in $\mathbb{R}^d$, H be some hyperplane of $\mathbb{R}^d$, and $\widetilde{A}$ be the Steiner symmetral of A with respect to H. If for some $u \in S^{d-1}$, $H = \{x \in \mathbb{R}^d; \langle x, u\rangle = 0\}$, let $P = \{x \in H; x + tu \in A$ for some $t \in \mathbb{R}\}$ be the image of A by the orthogonal projection onto H, and for $x \in P$, let $A(x) = \{t \in \mathbb{R}; x + tu \in A\}$ ($A(x)$ is a segment); then we have

$$\widetilde{A} = \left\{x + \lambda u; x \in P, \lambda \in \frac{1}{2}(A(x) - A(x))\right\}.$$

Given a convex body C in $\mathbb{R}^d$, symmetric with respect to H and such that $C \subset \widetilde{A}$, does there exist a convex body $B \subset A$ such that its Steiner symmetral $\widetilde{B}$ with respect to H satisfies $\widetilde{B} = C$? By Corollary 3 or 4, the answer is yes if $d = 2$; but Proposition 5 shows that it is generally no if $d \ge 3$.

One can ask, however, given the hyperplane H, what are the convex bodies A in $\mathbb{R}^d$, $d \geq 3$, such that the following property holds:

($*$) For every convex body $C \subset \widetilde{A}$, symmetric with respect to H, there exists a convex body $B \subset A$ such that $\widetilde{B} = C$.

If we suppose that A is smooth and strictly convex, we have the following answer:

Fact. If $A = \{(X, t); X \in P, t \in [-g_2(X), g_1(X)]\}$ as in the proof of Proposition 5 and if A is smooth and strictly convex and satisfies ($*$) with respect to the hyperplane $\mathbb{R}^{d-1} = \{t = 0\}$, then for some $\delta \geq 0$ and some affine function u on $\mathbb{R}^{d-1}$ one has

$$g_2 + u = \delta(g_1 - u) \quad \text{on } P.$$

Proof. By ($*$), since $0 \leq g = g_1 + g_2$, there exists an affine function u on $\mathbb{R}^{d-1}$ such that $-g_2 \leq u \leq g_1$ on P. Since A is strictly convex, we have $g_1 = g_2 = u$ on ∂P. Changing g_1 into $g_1 - u$ and g_2 into $g_2 + u$, we can suppose that $u = 0$, so that $g_1 = g_2 = 0$ on ∂P. Now let $X_1 \neq X_2$ in the interior of P, and let $M_j = (X_j, g(X_j))$ for $j = 1, 2$. We define a concave function f on P by $\{(X, t); X \in P, 0 \leq t \leq f(X)\} = \operatorname{conv}(M_1, M_2, \partial P)$. It follows from ($*$) that $f = h_1 + h_2$ for some concave functions h_i on P such that $0 \leq h_i \leq g_i$, $i = 1, 2$. Observe that $g_i(X_j) = h_i(X_j)$ for $i, j = 1, 2$. Let E be some affine hyperplane through M_1M_2 intersecting $\mathbb{R}^{d-1}$ on a tangent hyperplane to P in some point $Y \in \partial P$. Then it is clear that f and thus h_1 and h_2 must be affine on the triangle X_1X_2Y. Let F be the two-dimensional affine space generated by these points, $G = P \cap F$, and $N_j = (X_j, h_1(X_j))$, $j = 1, 2$. Since A is smooth, the convex body G has a unique tangent line at Y, which is $T = E \cap F$. It follows that either the lines M_1M_2, N_1N_2, and T are parallel or they intersect. Thus we get

$$\frac{(g_1 + g_2)(X_1)}{g_1(X_1)} = \frac{(g_1 + g_2)(X_2)}{g_1(X_2)}. \quad \square$$

In the preceding proof, the hypotheses of smoothness and of strict convexity of A can be replaced by the following assumptions: ∂A does not contain any nontrivial segment orthogonal to H and ∂P is smooth. We conjecture that our characterization of property ($*$) holds in the general case. It may be observed that if a convex body in $\mathbb{R}^d$, $d \geq 3$, satisfies ($*$) with respect to any hyperplane H, then it follows from the Kakutani theorem that A is an ellipsoid.

Remark. Let A be given as in Corollary 3, and suppose 0 is in the interior of D. Then $A = \bigcup_{t \in [0, 1]}(h(t) + r(t)D, t)$ for some concave positive function $r: [0, 1] \to \mathbb{R}_+$ and some function $h: [0, 1] \to \mathbb{R}^{d-1}$. We proved in Corollary 3 that given a concave function $s: [0, 1] \to \mathbb{R}_+$ such that $0 \leq s \leq r$, there exists a function $k: [0, 1] \to \mathbb{R}^{d-1}$ such that $B = \bigcup_{t \in [0, 1]}(k(t) + s(t)D, t)$ is a convex body in $\mathbb{R}^d$ contained in A. One can ask also if, given a function $k: [0, 1] \to \mathbb{R}^{d-1}$, one can find some function $s: [0, 1] \to \mathbb{R}_+$ such that $B = \bigcup_{t \in [0, 1]}(k(t) + s(t)D, t)$ satisfies the same conclusion. It can be proved that the answer is positive if and only if

(a) If $k(t) = (k_1(t), \ldots, k_{d-1}(t))$, then the k_i, $i = 1, \ldots, d-1$, are continuous and, as distributions, have second derivative k_i'' which are measures on $[0, 1]$.

(b) There exists α, $\beta \in \mathbb{R}$ such that for every $x \in [0, 1]$,

$$N_D(k(x) - h(x)) + \alpha x + \beta + \int_0^1 G(x, y)\, d\mu(y) \le r(x),$$

where $N_D\colon \mathbb{R}^{d-1} \to \mathbb{R}_+$ is defined by $N_D(y) = \inf\{\lambda > 0;\, y \in \lambda D\}$, μ is the total variation measure on $[0, 1]$, with respect to N_D, of the vector measure $-k'' = (-k_1'', \ldots, -k_{d-1}'')$, and G is the classical Green function of $[0, 1]$: $G(x, y) = x(1-y)$ if $0 \le x \le y \le 1$ and $G(x, y) = y(1-x)$ if $0 \le y \le x \le 1$. Observe that if the k_i'' have continuous density f_i with respect to the Lebesgue measure dy on $[0, 1]$, then

$$\int_0^1 G(x, y)\, d\mu(y) = \int_0^1 N_D((-f_i(x))_{i=1}^{d-1}) G(x, y)\, dy.$$

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