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AN ANSWER TO A CONJECTURE ON THE COUNTABLE PRODUCTS OF k-SPACES

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ABSTRACT. In this paper the author shows:

Theorem (CH). *There is a k_ω-space X which is not locally compact but for which X^ω has a k-system.*

This answers a conjecture of Y. Tanaka

1. INTRODUCTION

Let $\mathscr{C}$ be a compact covering of a space X. Then X is said to have the weak topology with respect to $\mathscr{C}$ if $U \subset X$ is open (closed) in X whenever $U \cap C$ is open (closed) in C for each $C \in \mathscr{C}$. Following Arhangelskii [1] such a covering is called a k-system. We recall that a space X is a k-space if it has the weak topology with respect to the cover consisting of all compact subsets of X. Then a space with a k-system is precisely a k-space. If $\mathscr{C}$ is point-countable (countable), X is said to have a point-countable k-system (be a k_ω-space). Recall a space X has countable tightness $(t(X) \leq \omega)$ if whenever $x \in \overline{A}$, then $x \in \overline{C}$ for some countable $C \subset A$. Arhangelskii [2] has shown that X^ω has a k-system if X is a locally compact space. Y. Tanaka has shown that if X has a point-countable k-system and $t(X) \leq \omega$, then X^ω has a k-system if and only if X is locally compact. In [3–5] Tanaka posed the following question: Let X have a (point-) countable k-system. Then is X locally compact if X^ω has a k-system? In this paper we give a negative answer under (CH). Namely, under (CH), there is a k_ω-space X which is not locally compact, but X^ω has a k-system.

2. RESULTS

For the ordinal $\omega_2 + 1$, let Z be the topological sum of countably many copies of $[0, \omega_2]$ with the order topology. Then let X be the quotient space obtained from Z by identifying all the ω_2's. Let $f: Z \to X$ be a canonical projection. Then f is a closed mapping and X is a k_ω-space. Let $\mathscr{C} = \{C_n = \bigcup_{i \leq n}[0, \omega_2]_i ; n < \omega\}$; then $\mathscr{C}$ is a countable k-system of Z. It

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will not cause confusion if we denote a k-system of X by $\mathscr{C}$. Let $\mathscr{C}^\omega = \{\prod_{i<\omega} C_{n_i}; C_{n_i} \in \mathscr{C}\}$; then $\mathscr{C}^\omega$ is a k-system of the space Z^ω and $\mathscr{C}^\omega$ is a compact covering of the space X^ω. $f: Z \to X$ is a continuous mapping, and so is $f^\omega = \prod_{i<\omega} f_i: Z^\omega \to X^\omega$, where $f_i = f$.

Lemma 1. *Suppose C is a compact subset in X^ω or Z^ω. Then X^ω or Z^ω has a $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$ such that $C \subset \prod_{i<\omega} C_{n_i}$.*

Proof. By [4, Lemma 6].

Lemma 2. *Let $f^\omega: Z^\omega \to X^\omega$ be a continuous mapping. For any $U \subset X$, $(f^\omega)^{-1}(U)$ is open in Z^ω iff $(\prod_{i<\omega} C_{n_i}) \cap U$ is open in $\prod_{i<\omega} C_{n_i}$ for each $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$.*

Proof. Suppose $(\prod_{i<\omega} C_{n_i}) \cap U$ is open in $\prod_{i<\omega} C_{n_i}$ for each $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$ and $f^\omega | \prod_{i<\omega} C_{n_i}: \prod_{i<\omega} C_{n_i} \to \prod_{i<\omega} C_{n_i}$ is continuous. Let

$$A = \left(f^\omega | \prod_{i<\omega} C_{n_i}\right)^{-1} \left[\left(\prod_{i<\omega} C_{n_i}\right) \cap U\right] = (f^\omega)^{-1}(U) \cap \left(\prod_{i<\omega} C_{n_i}\right);$$

then A is open in $\prod_{i<\omega} C_{n_i}$ $(\subset Z^\omega)$. Since Z^ω has a k-system $\mathscr{C}^\omega$ by Lemma 1, $(f)^{-1}(U)$ is open in Z^ω.

Now assume $(f^\omega)^{-1}(U)$ is open in Z^ω. Then A is open in $\prod_{i<\omega} C_{n_i}$ for each $\prod_{i<\omega} C_{n_i}$. Therefore $\prod_{i<\omega} C_{n_i} - A$ is compact in $\prod_{i<\omega} C_{n_i}$, and so is $f^\omega[\prod_{i<\omega} C_{n_i} - A] = \prod_{i<\omega} C_{n_i}[[(\prod_{i<\omega} C_{n_i}) \cap U]$. Then $U \cap (\prod_{i<\omega} C_{n_i})$ is open in $\prod_{i<\omega} C_{n_i}$.

Theorem (CH). *The space X is a k_ω-space which is not locally compact. But X^ω has a k-system.*

Proof. By Lemma 1 it suffices to prove that given $U \subset X^\omega$, if $U \cap (\prod_{i<\omega} C_{n_i})$ is open in $\prod_{i<\omega} C_{n_i}$ for each $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$, then U is open in X^ω. Let $x \in U$; the general proof follows from the next three cases.

(A) $x = (\alpha_i)$, $\alpha_i \neq \omega_2$, $i < \omega$. $(f^\omega)^{-1}(\alpha_i) = (\alpha_i) \in (f^\omega)^{-1}(U)$. By Lemma 2 $(f^\omega)^{-1}(U)$ is open in Z^ω. Then (α_i) has a neighborhood $W = [\delta_1, \alpha_1'] \times \cdots \times [\delta_n, \alpha_n'] \times Z \times Z \times \cdots$ in Z^ω such that $(\alpha_i) \in W \subset (f^\omega)^{-1}(U)$. Then $(\alpha_i) \in f^\omega(W) = [\delta_1, \alpha_1'] \times \cdots \times [\delta_n, \alpha_n'] \times X \times X \times \cdots \subset U$. Hence U is a neighborhood of the point x in X^ω.

(B) $x = (\omega_2, \omega_2, \dots) = (\omega_2)$. First we prove that there is an $N < \omega$ such that for each $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$, there is a $\delta \in [0, \omega_2)$ with $(\bigcup_{i \le n_1} [\delta, \omega_2]_i) \times (\bigcup_{i \le n_2} [\delta, \omega_2]_i) \times \cdots \times (\bigcup_{i \le n_N} [\delta, \omega_2]_i) \times \prod_{i>N} C_{n_i} \subset U$.

Suppose that for each $n < \omega$ there is a $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$ such that for each $\delta \in [0, \omega_2)$, $[(\bigcup_{i \le n_1} [\delta, \omega_2]_i) \times \cdots \times (\bigcup_{i \le n_n} [\delta, \omega_2]_i) \times \prod_{i>n} C_{n_i}] - U \neq \varnothing$.

Then there is a $y_\delta = (\delta_1', \delta_2', \dots, \delta_n', x_{\delta 1}, x_{\delta 2}, \dots)$ such that

$$y_\delta \in \left[\left(\bigcup_{i \le n_1} [\delta, \omega_2]_i\right) \times \cdots \times \left(\bigcup_{i \le n_n} [\delta, \omega_2]_i\right) \times \prod_{i>n} C_{n_i}\right] - U.$$

Let $B_n = \{(\delta_1', \delta_2', \dots, \delta_n', x_{\delta 1}, x_{\delta 2}, \dots); \delta < \omega_2\}$. Then $B_n \cap U = \varnothing$ and $B_n \subset \prod_{i<\omega} C_{n_i}$. We show $\overline{B}_n \cap (\{\omega_2\} \times \{\omega_2\} \times \cdots \times \{\omega_2\} \times \prod_{i>n} C_{n_i}) \neq \varnothing$. In fact, suppose $\overline{B}_n \cap (\{\omega_2\} \times \{\omega_2\} \times \cdots \times \{\omega_2\} \times \prod_{i>n} C_{n_i}) = \varnothing$. Since $\overline{B}_n$ and $\{\omega_2\} \times \{\omega_2\} \times \cdots \times \{\omega_2\} \times \prod_{i>n} C_{n_i}$ are compact in $\prod_{i<\omega} C_{n_i}$, $\{\omega_2\} \times \cdots \times \{\omega_2\} \times$

$\prod_{i>n} C_{n_i}$ has a neighborhood $W = (\bigcup_{i\le n_1}[\delta_0, \omega_2]_i) \times \cdots \times (\bigcup_{i\le n_n}[\delta_0, \omega_2]_i) \times \prod_{i>n} C_{n_i}$ in $\prod_{i<\omega} C_{n_i}$ such that $W \cap \overline{B}_n = \varnothing$. On the other hand, we let

$$A_n = \{(\delta_1', \delta_2', \dots, \delta_n', x_{\delta 1}, x_{\delta_2}, \dots) \in B_n ; \delta_0 \le \delta < \omega_2\};$$

then $A_n \ne \varnothing$, $A_n \subset B_n$, and $A_n \subset W$. Therefore, $A_n = A_n \cap B_n \subset W \cap \overline{B}_n = \varnothing$, a contradiction.

Hence there is a $(\omega_2, \omega_2, \dots, \omega_2, x_{n1}, x_{n2}, \dots) \in \overline{B}_n$. Since $(\prod_{i<\omega} C_{n_i}) \cap U$ is open in $\prod_{i<\omega} C_{n_i}$, $(\omega_2, \omega_2, \dots, \omega_2, x_{n1}, x_{n2}, \dots) \in \overline{B}_n$ and $\overline{B}_n \subset \prod_{i<\omega} C_{n_i} - [(\prod_{i<\omega} C_{n_i}) \cap U]$; hence, $(\omega_2, \omega_2, \dots, \omega_2, x_{n1}, x_{n2}, \dots) \notin U$. Let

$$B = \{(\omega_2, x_{11}, x_{12}, \dots), (\omega_2, \omega_2, x_{21}, x_{22}, \dots), \dots, (\omega_2, \dots, \omega_2, x_{n1}, \dots), \dots\};$$

then $B \cap U = \varnothing$. $\prod_1(B) = \{\omega_2\}$, $\prod_{i+1}(B) = \{\omega_2, x_{1i}, x_{2i-1}, \dots, x_{i1}\}$, $i = 1, 2, \dots$. For each $i < \omega$, $\prod_i(B)$ is finite in X, so there is a $C_{m_i} \in \mathscr{C}$ such that $B \subset \prod_{i<\omega} C_{m_i}$. Since $U \cap (\prod_{i<\omega} C_{m_i})$ is open in $\prod_{i<\omega} C_{m_i}$, we have $\overline{B} \subset \prod_{i<\omega} C_{m_i} - [(\prod_{i<\omega} C_{m_i}) \cap U]$. This B is a sequence converging to $(\omega_2, \omega_2, \dots)$ in $\prod_{i<\omega} C_{m_i}$. Indeed, take a neighborhood $W = (\bigcup_{i\le m_1}[\delta, \omega_2]_i) \times \cdots \times (\bigcup_{i\le m_n}[\delta, \omega_2]_i) \times \prod_{i>n} C_{m_i}$ of $(\omega_2, \omega_2, \dots)$ in $\prod_{i<\omega} C_{m_i}$. Then $\{(\omega_2, \dots, \omega_2, x_{n+1,1}, x_{n+1,2}, \dots), (\omega_2, \dots, \omega_2, x_{n+2,1}, x_{n+2,2}, \dots), \dots\} \subset W$. Therefore, $(\omega_2, \omega_2, \dots) \in \overline{B} \subset \prod_{i<\omega} C_{m_i} - [(\prod_{i<\omega} C_{m_i}) \cap U]$, $(\omega_2) \notin U$. It is a contradiction with $(\omega_2) \in U$. Second we look for a neighborhood W of (ω_2) in X^ω such that $(\omega_2) \in W \subset U$. Let $\mathscr{C}^\omega = \{\prod_{i<\omega} C_{n_i} ; C_{n_i} \in \mathscr{C}\} = \{K_\alpha ; \alpha < \omega_1\}$. By the previous arguments, there is an $N < \omega$ such that for each $K_\alpha = \prod_{i<\omega} C_{n_i}$ there is a $\delta_\alpha < \omega_2$ such that $(\bigcup_{i\le n_1}[\delta_\alpha, \omega_2]_i) \times \cdots \times (\bigcup_{i\le n_N}[\delta_\alpha, \omega_2]_i) \times \prod_{i>N} C_{n_i} \subset U$. Let $\delta = \sup_{\alpha<\omega_1} \delta_\alpha$; then $\delta < \omega_2$ and $[(\bigcup_{i\le n_1}[\delta, \omega_2]_i) \times \cdots \times (\bigcup_{i\le n_N}[\delta, \omega_2]_i) \times \prod_{i>N} C_{n_i} \subset U$ for any $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$. Thus let $V = \bigcup_{i<\omega}[\delta, \omega_2]_i$ be a neighborhood of the point (ω_2) in X; then

$$\bigcup_{\alpha<\omega_1} \left[\left(\bigcup_{i\le n_1} [\delta, \omega_2]_i \right) \times \cdots \times \left(\bigcup_{i\le n_N} [\delta, \omega_2]_i \right) \times \prod_{i>N} C_{n_i} \right.$$
$$= V \times \cdots \times V \times X \times X \times \cdots = W \subset U.$$

(C) $x = (\alpha_1, \omega_2, \alpha_2, \omega_2, \dots, \alpha_n, \omega_2, \alpha_{n+1}, \omega_2, \dots)$, $\alpha_i \ne \omega_2$. First we show that there is an $N < \omega$ and a $\delta_i \in (0, \alpha_i)$, $i = 1, 2, \dots, N$, such that for each $\delta_i' \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, N$, there is a $V \in \mathscr{U}$ with $\{\delta_1'\} \times V \times \{\delta_2'\} \times V \times \cdots \times \{\delta_n'\} \times V \times X \times X \times \cdots \subset U$, where $\mathscr{U} = \{V_\alpha = \bigcup_{i<\omega}[\alpha, \omega_2]_i ; \alpha < \omega_2\}$. Suppose for each $n < \omega$ and each $\delta_i \in (0, \alpha_i)$, $i = 1, 2, \dots, n$, there is a $\delta_i' \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, n$, such that for each $V \in \mathscr{U}$, $(\{\delta_1'\} \times V \times \{\delta_2'\} \times V \times \cdots \times \{\delta_n'\} \times V \times X \times X \times \cdots) - U \ne \varnothing$. Take any $\delta_i \in (0, \alpha_i)$, $i < \omega$. For each $n < \omega$ and each $\delta_i \in (0, \alpha_i)$, $i = 1, 2, \dots, n$, there is a $\delta_i' \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, n$, such that for each $V \in \mathscr{U}$ there is an $x_v = (\delta_{n1}, \alpha_{v1}, \delta_{n2}, \alpha_{v2}, \dots, \delta_{nn}, \alpha_{vn}, x_{v1}, x_{v2}, \dots)$ such that $x_v \in (\{\delta_1'\} \times V \times \{\delta_2'\} \times V \times \cdots \times \{\delta_n'\} \times V \times X \times X \times \cdots) - U$. Let $B_n = \{x_v = (\delta_{n1}, \alpha_{v1}, \delta_{n2}, \alpha_{n2}, \dots, \delta_{nn}, \alpha_{vn}, x_{v1}, x_{v2}, \dots) ; V \in \mathscr{U}\}$. There are two possible cases.

Case 1. $|B_n| < \aleph_2$. By $|\mathscr{U}| = \aleph_2$ there is an $x_{v'}$ which repeats at least $\aleph_2$ times in B_n. Then the point $x_{v'} = (\beta_1, \omega_2, \beta_2, \omega_2, \dots, \beta_n, \omega_2, x_{n1}, x_{n2}, \dots)$,

$x_{n_i} \in X$, $i < \omega$, and $\beta_i \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, n$. If some $2k$th $(k \le n)$ coordinate of $x_{v'}$ is not ω_2, then let this $2k$th $(k \le n)$ coordinate be α and $\mathscr{U}_1 = \{V \in \mathscr{U}; V \supset V_{\alpha+1}\}$. Since $\mathscr{U}$ is linearly ordered by inclusion, $|\mathscr{U}_1| \le \aleph_1$; therefore, $x_{v'}$ repeats at most $\aleph_1$ times in $\{x_v \in B_n; V \in \mathscr{U}_1\}$. Hence there is a $\delta \ge \alpha+1$ such that $x_{v_\delta} = x_{v'}$. On the other hand, if $\alpha \in V_\delta = \bigcup_{i<\omega}[\delta, \omega_2]_i$, then $\alpha \ge \delta \ge \alpha+1$, which is a contradiction. By the $x_{v'} \in B_n$ and $B_n \cap U = \varnothing$, it follows that $x_{v'} \notin U$.

Case 2. $|B_n| = \aleph_2$. Since $|\{\prod_{i<\omega} C_{n_i}; C_{n_i} \in \mathscr{C}\}| = \aleph_1$, there is a $\prod_{i<\omega} C_{n_i} \in \mathscr{C}^\omega$ such that $|(\prod_{i<\omega} C_{n_i}) \cap B_n| = \aleph_2$. Let $\mathscr{U}_1 = \{V \in \mathscr{U}; x_v \in B_n \cap (\prod_{i<\omega} C_{n_i})\}$; then $\mathscr{U}_1$ is confinal in $\mathscr{U}$. Let $A_n = \{x_v \in B_n \cap (\prod_{i<\omega} C_{n_i}); V \in \mathscr{U}_1\}$ (hence $A_n \subset \prod_{i<\omega} C_{n_i}$) and $K = \{\alpha_1\} \times \{\omega_2\} \times \{\alpha_2\} \times \{\omega_2\} \times \cdots \times \{\alpha_n\} \times \{\omega_2\} \times \prod_{i>2n} C_{n_i}$; then we will show that $\overline{A}_n \cap K \ne \varnothing$. Indeed, suppose $\overline{A}_n \cap K = \varnothing$, take a neighborhood $W = [\beta_1', \alpha_1'] \times (\bigcup_{i \le n_2}[\delta, \omega_2]_i) \times \cdots \times [\beta_n', \alpha_n'] \times (\bigcup_{i \le n_{2n}}[\delta, \omega_2]_i) \times \prod_{i>2n} C_{n_i}$ of K in $\prod_{i<\omega} C_{n_i}$ such that $W \cap \overline{A}_n = \varnothing$. Let $A_n^* = \{x_v \in A_n; V \subset V_\delta = \bigcup_{i<\omega}[\delta, \omega_2]_i\}$; then $A_n^* \subset W$ and $|A_n^*| = \aleph_2$. On the other hand, $A_n^* \subset A_n$, so $A_n^* = A_n^* \cap A_n \subset W \cap A_n = \varnothing$, a contradiction. Thus there is an $x_{n_i} \in C_{n_{2n+i}}$ $(i < \omega)$ such that

$$(\alpha_1, \omega_2, \alpha_2, \omega_2, \dots, \alpha_n, \omega_2, x_{n1}, x_{n2}, \dots) \in \overline{A}_n \subset \prod_{i<\omega} C_{n_i} - \left[\left(\prod_{i<\omega} C_{n_i}\right) \cap U\right].$$

Hence $(\alpha_1, \omega_2, \alpha_2, \omega_2, \dots, \alpha_n, \omega_2, x_{n1}, x_{n2}, \dots) \notin U$. Therefore, for either Case 1 or 2, there is a $\delta_{n_i} \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, n$, such that

$$y_n = (\delta_{n1}, \omega_2, \delta_{n2}, \omega_2, \dots, \delta_{nn}, \omega_2, x_{n1}, x_{n2}, \dots) \notin U.$$

Let $B = \{y_n; n = 1, 2, \dots\}$. Then $\prod_1(B) = \{\delta_{1+i,1}; i < \omega\}$, $\prod_2(B) = \{\omega_2\}$, $\prod_{2n+1}(B) = \{x_{1,2n-1}, x_{2,2n-3}, \dots, x_{n,1}\} \cup \{\delta_{n+1+i,n+1}; i < \omega\}$, $n = 1, 2, \dots$, $\prod_{2n}(B) = x_{1,2n-2}, x_{2,2n-4}, \dots, x_{n-1,2}\} \cup \{\omega_2\}$, $n = 2, 3, \dots$, and $B \cap U = \varnothing$. Obviously, for each $i < \omega$ there is a $C_{m_i} \in \mathscr{C}$ such that $\prod_i(B) \subset C_{m_i}$; therefore, $B \subset \prod_{i<\omega} C_{m_i}$. Let $K = (\delta_1, \alpha_1] \times \{\omega_2\} \times \cdots \times (\delta_n, \alpha_n] \times \{\omega_2\} \times (\delta_{n+1}, \alpha_{n+1}] \times \{\omega_2\} \times \cdots$; then $K \cap \overline{B} \ne \varnothing$. In fact, suppose $\overline{B} \cap K = \varnothing$, where we assume $K \subset \prod_{i<\omega} C_{m_i}$. Then there is a neighborhood $W = [B_1, \alpha_1] \times (\bigcup_{i \le m_2}[\delta, \omega_2]_i) \times \cdots \times [\beta_n, \alpha_n] \times (\bigcup_{i \le m_n}[\delta, \omega_2]_i) \times \prod_{i>n} C_{m_i}$ of K in $\prod_{i<\omega} C_{m_i}$ such that $W \cap \overline{B} = \varnothing$. On the other hand, $\varnothing \ne \{y_n, y_{n+1}, \dots\} \subset B \cap W = \varnothing$, a contradiction. Take any $x \in K \cap \overline{B} \ne \varnothing$, $x = (\delta_1', \omega_2, \delta_2', \omega_2, \dots, \delta_n', \omega_2, \dots)$. By $U \cap (\prod_{i<\omega} C_{m_i})$ being open in $\prod_{i<\omega} C_{m_i}$ and $B \subset \prod_{i<\omega} C_{m_i} - [(\prod_{i<\omega} C_{m_i}) \cap U]$, it follows that $(\delta_1', \omega_2, \dots, \delta_n', \omega_2, \delta_{n+1}', \omega_2, \dots) \in \overline{B} \subset \prod_{i<\omega} C_{m_i} - [(\prod_{i<\omega} C_{m_i}) \cap U]$. Therefore, $(\delta_1', \omega_2, \dots, \delta_n', \omega_2, \delta_{n+1}', \omega_2, \dots) \notin U$. Hence it is proven that for each $\delta_i \in (0, \alpha_i)$, $i < \omega$, there is a $\delta_i' \in (\delta_i, \alpha_i]$, $i < \omega$, such that $(\delta_1', \omega_2, \dots, \delta_n', \omega_2, \delta_{n+1}', \omega_2, \dots) \notin U$. Let

$$A = \{(\delta_1', \omega_2, \dots, \delta_n', \omega_2, \dots); \delta_i' \in (0, \alpha_i), \delta_i' \in (\delta_i, \alpha_i], i < \omega\};$$

then $A \cap U = \varnothing$ and $A \subset K_1 = [0, \alpha_1] \times \{\omega_2\} \times \cdots \times [0, \alpha_n] \times \{\omega_2\} \times [0, \alpha_{n+1}] \times \{\omega_2\} \times \cdots$. By $U \cap K_1$ being open in K_1, it follows that $(\alpha_1, \omega_2, \dots, \alpha_n, \omega_2, \dots) \in \overline{A} \subset K_1 - (K_1 \cap U)$. This contradicts the fact

$(\alpha_1, \omega_2, \dots, \alpha_n, \omega_2, \alpha_{n+1}, \omega_2, \dots) \in U$. Second, we look for a neighborhood W of $(\alpha_1, \omega_2, \dots, \alpha_n, \omega_2, \dots)$ in X^ω such that $W \subset U$. We already showed that there is an $N < \omega$ and a $\delta_i \in (0, \alpha_i)$, $i = 1, 2, \dots, N$, such that for each $\delta_i' \in (\delta_i, \alpha_i]$, $i = 1, 2, \dots, N$, there exists a $V \in \mathscr{U}$ with $\{\delta_1'\} \times V \times \{\delta_2'\} \times V \times \cdots \times \{\delta_N'\} \times V \times X \times X \times \cdots \subset U$. On the other hand,

$$|[0, \alpha_1] \times [0, \alpha_2] \times \cdots \times [0, \alpha_n]| \leq \aleph_1 < \aleph_2.$$

Then for each $(\delta_1', \delta_2', \dots, \delta_N') \in (\delta_1, \alpha_1] \times \cdots \times (\delta_N, \alpha_N]$, there is an $\alpha < \omega_2$ such that $\{\delta_1'\} \times V_\alpha \times \cdots \times \{\delta_N'\} \times V_\alpha \times X \times X \times \cdots \subset U$. Let $\delta = \sup_{\alpha<\omega_1} \alpha$. Then $\delta < \omega_2$, $V_\delta \in \mathscr{U}$, and for each $(\delta_1', \delta_2', \dots, \delta_N') \in (\delta_1, \alpha_1] \times \cdots \times (\delta_N, \alpha_N]$, $\{\delta_1'\} \times V_\delta \times \cdots \times \{\delta_N'\} \times V_\delta \times X \times X \times \cdots \subset U$. Therefore, $(\delta_1, \alpha_1] \times V_\delta \times \cdots \times (\delta_N, \alpha_N] \times V_\delta \times X \times X \times \cdots \subset U$.

(D) In general, if $x \in U$, we can show that U is a neighborhood of x in X^ω by using the technique of (B) or (C).

Remark. In the previous theorem, we can replace "not locally compact" by "not even a k'-space". Indeed, let $Z_0 = [1, \omega_0]$, $Z_n = [0, \omega_2]_n$ for $n = 1, 2, \dots$, with order topology. Let Z be the topological sum of Z_n, $i < \omega$; then Z is a locally compact Hausdorff space. Let Y be the quotient space obtained from Z by identifying each "n" in Z_0 and "ω_2" in Z_n for $n = 1, 2, \dots$. Then Y is a k_ω-space but not a k'-space. In fact, $\omega_0 \in \overline{\bigcup_{i<\omega}[0, \omega_2)_i}$ but $\omega_0 \notin \overline{K \cap (\bigcup_{n<\omega}[0, \omega_2)_n)}$ for each compact subset $K \subset Y$. Let $g: Y \to Y/[1, \omega_0] = X$ be a canonical projection. Then g is a prefect mapping, as is g^ω. Since X^ω has a k-system, so does Y^ω. Then Y has the desired property.

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