

A DUALITY PROOF OF A THEOREM OF P. HILL

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(Communicated by Ronald Solomon)

ABSTRACT. A duality argument is given to prove the equivalence of a recent theorem of P. Hill and the main step in Zippin's proof of Ulm's theorem.

Let G and H be isomorphic finite abelian p -groups with isomorphic subgroups A and B , respectively, and set $X = G/A$, $Y = H/B$. In his 1935 proof of Ulm's theorem, Zippin established the following result.

Theorem (Zippin [2, p. 30], [3]). *If $\varphi : A \rightarrow B$ is an isomorphism, then there exists an isomorphism $G \rightarrow H$ that induces φ if and only if φ preserves heights.*

Recently, P. Hill has proved the following theorem (stated in a slightly different but equivalent form).

Theorem (P. Hill [1]). *If $\psi : X \rightarrow Y$ is an isomorphism, then there exists an isomorphism $G \rightarrow H$ that induces ψ if and only if ψ preserves coset covalues.*

Hill's theorem appears to be dual to Zippin's theorem. The purpose of this note is to make the connection between these two results more explicit by giving a duality proof of their equivalence.

First, recall the definitions. Let P be any finite abelian p -group and let $0 \neq x \in P$. The *height* of x in P is the unique integer $k \geq 0$ for which $x \in p^k P \setminus p^{k+1} P$ (and the height of 0 is ∞ , by convention). Similarly, the *exponent* of x in P is the unique integer $k > 0$ for which $x \in P[p^k] \setminus P[p^{k-1}]$ where $P[p^k] := \{y \in P \mid p^k y = 0\}$ (and the exponent of 0 is zero). If $S \leq P$, then the *coset covalue* of $x + S$ is by definition the least among the exponents of the elements of $x + S$.

With the notation as above, set $A_k = A \cap p^k G$ and $X^k = G/(A + G[p^k])$, and define B_k and Y^k similarly. Let $\pi_X : G \rightarrow X$ and $\pi_Y : G \rightarrow Y$ denote the canonical epimorphisms.

Lemma. (i) *An isomorphism $A \rightarrow B$ preserves heights if and only if it induces an isomorphism $A_k \rightarrow B_k$ for each k .*

(ii) *An isomorphism $X \rightarrow Y$ preserves coset covalues if and only if it induces an isomorphism $X^k \rightarrow Y^k$ for each k .*

Proof. The proof of (i) is trivial.

Received by the editors May 5, 1993.

1991 *Mathematics Subject Classification.* Primary 20K30; Secondary 20E07.

(ii) We can make the identifications

$$X^k = X/\pi_X(G[p^k]) \quad \text{and} \quad Y^k = Y/\pi_Y(H[p^k])$$

so that an isomorphism $X \rightarrow Y$ induces an isomorphism $X^k \rightarrow Y^k$ if and only if it induces an isomorphism $\pi_X(G[p^k]) \rightarrow \pi_Y(H[p^k])$. The statement now follows since clearly the coset covalue of a coset $g + A$ is k if and only if $g + A \in \pi_X(G[p^k]) \setminus \pi_X(G[p^{k-1}])$. \square

In view of this lemma, the equivalence of Zippin's theorem and Hill's theorem can be stated as follows.

Theorem. *The following statements are equivalent.*

- (Z) *If $\varphi : A \rightarrow B$ is an isomorphism, then there exists an isomorphism $G \rightarrow H$ that induces φ if and only if φ induces an isomorphism $A_k \rightarrow B_k$ for each k .*
- (H) *If $\psi : X \rightarrow Y$ is an isomorphism, then there exists an isomorphism $G \rightarrow H$ that induces ψ if and only if ψ induces an isomorphism $X^k \rightarrow Y^k$ for each k .*

We need some preliminaries before proving the theorem. Denote the contravariant functor $\text{Hom}(\cdot, \mathbb{Z}(p^\infty))$ on finite abelian p -groups by $P \mapsto P^*$ (object map) and $\varphi \mapsto \varphi^*$ (morphism map). This functor is exact and P^{**} is naturally isomorphic to P (cf. [2, pp. 79–80]). As usual, if $S \leq P$, we identify S^* with P^*/S^0 and $(P/S)^*$ with S^0 where $S^0 = \{f \in P^* \mid f(S) = 0\}$.

Lemma 1. *Let $\eta : G \rightarrow H$ be a homomorphism.*

- (i) *If η induces a homomorphism $\psi : X \rightarrow Y$, then η^* induces ψ^* .*
- (ii) *If η induces a homomorphism $\varphi : A \rightarrow B$, then η^* induces φ^* .*

Proof. (i) Assume η induces a homomorphism $\psi : X \rightarrow Y$. This means that $\psi \circ \pi_X = \pi_Y \circ \eta$. Applying $*$, we have $\pi_X^* \circ \psi^* = \eta^* \circ \pi_Y^*$. Since $\pi_X^* : X^* \rightarrow G^*$ and $\pi_Y^* : Y^* \rightarrow H^*$ are the inclusion maps, this equation says that η^* induces ψ^* .

(ii) Similar. \square

Lemma 2. (i) $A_k^* = A^{*k}$.

(ii) $X^{k*} = X^*_{*k}$.

Proof. (i) $A_k^* = G^*/A_k^0 = G^*/(A^0 + (p^k G)^0) = A^{*k}$, since $(p^k G)^0 = G^*[p^k]$.

(ii) Use (i) with $X^* \leq G^*$ in place of $A \leq G$ and then apply $*$. \square

Proof of Theorem. (Z) \Rightarrow (H). Assume that statement (Z) holds. One implication in (H) is trivial. For the other, suppose we are given an isomorphism $\psi : X \rightarrow Y$ that induces an isomorphism $X^k \rightarrow Y^k$ for each k . By Lemma 1(i) and Lemma 2(ii), the isomorphism $\psi^* : Y^* \rightarrow X^*$ induces an isomorphism $Y^*_{*k} \rightarrow X^*_{*k}$ for each k . Part (Z) of the theorem applies to give an isomorphism $\eta : H^* \rightarrow G^*$ that induces ψ^* . Using Lemma 1(ii) and natural isomorphisms, we obtain an isomorphism $G \rightarrow H$ that induces ψ .

(H) \Rightarrow (Z). Similar. \square

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