PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

LINEAR TAME EXTENSION OPERATORS FROM CLOSED SUBVARIETIES OF $\mathbb{C}^d$*

AYDIN AYTUNA

(Communicated by Theodore W. Gamelin)

Dedicated to Professor Walter Rudin

ABSTRACT. In this note we show that a linear continuous and tame extension operator from the space of analytic functions on a closed irreducible subvariety V of $\mathbb{C}^d$ exists if and only if V is an algebraic variety.

Let M be a Stein space. We will denote by $\mathscr{O}(M)$ the Fréchet space of analytic functions on M equipped with the topology of uniform convergence on compacta. If V is a closed subvariety of M, the question as to whether one can find a continuous linear extension operator from $\mathscr{O}(V)$ into $\mathscr{O}(M)$ was studied by various authors. For example, if V is a closed subvariety of $\mathbb{C}^d$, a continuous linear extension operator exists iff every bounded plurisubharmonic function on V reduces to a constant (see [12]), and this in turn is equivalent to the statement that $\mathscr{O}(V)$ and $\mathscr{O}(\mathbb{C}^k)$ are isomorphism as Fréchet spaces, $d = \dim V$; see [2] (cf. [1, 7, 11]). In this note we take up the question of existence of continuous extension operators from subvarieties of $\mathbb{C}^d$, in the category of graded Fréchet spaces and linear tame operators.

Recall that a *graded* Fréchet space $\{X, \| \|_k\}$ is a Fréchet space X with a fixed fundamental system of seminorms $\{\| \|_k\}$ generating its topology. A linear operator T from a graded Fréchet space $\{X_0, \| \|_k\}$ into a graded Fréchet space $\{X_1, | |_k\}$ is called a *linear tame* operator in case there exist positive constants A and B such that

$$\forall k\ \exists C_k > 0 \text{ such that } |T(x)|_k \le C_k \|x\|_{Ak+B} \quad \forall x \in X_0 .$$

For further information about this category and its relation to implicit function theorems in Fréchet spaces we refer the reader to [4] (cf. [10]).

In the Fréchet space $\mathscr{O}(\mathbb{C}^d)$ we consider the grading

$$\|F_n\| \doteq \sup_{z\in\overline{D}_n} |F(z)| \quad \forall F \in \mathscr{O}(\mathbb{C}^d),\ n = 1, 2,$$

Received by the editors January 4, 1993 and, in revised form, May 26, 1993.
1991 *Mathematics Subject Classification*. Primary 46E10.
Key words and phrases. Linear tame extension operators for analytic functions defined on subvarieties of $\mathbb{C}^d$, algebraic varieties.
This work was partially supported by the Scientific and Technical Research Council of Turkey (TBAG-Ç2).

where

$$D_n = \{z \in \mathbb{C}^d\,,\ \|z\| < e^n\}\,, \qquad n = 1\,,\,2\,,\,\ldots\,.$$

Moreover, if V is a closed subvariety of $\mathbb{C}^d$, then we will always consider on $\mathscr{O}(V)$ the grading

$$|f|_n \doteq \sup_{z\in K_n} |f(z)| \quad \forall f \in \mathscr{O}(V)\,,$$

where for n sufficiently large $K_n = \overline{D}_n \cap V$. We begin by recalling a result of Djakov and Mitiagin on the structure of polynomial ideals (see [3]).

Theorem 1. *Let V be an algebraic variety in $\mathbb{C}^d$. Then there exist polynomials $Q_1\,,\,\ldots\,,\,Q_p$ that generate the ideal*

$$I^*(V) = \{P \in \mathbb{C}[z_1\,,\,\ldots\,,\,z_d];\ P|V \equiv 0\}\,,$$

a vector $B = (b_1\,,\,\ldots\,,\,b_d)$, and continuous linear operators

$$R_i\colon \mathscr{O}(\mathbb{C}^d) \to \mathscr{O}(\mathbb{C}^d)\,, \qquad 0 \le i \le \rho\,,$$

such that
(i) $f = R_0(f) + \sum_{i=1}^{\rho} R_i(f)Q_i$ *for all* $f \in \mathscr{O}(\mathbb{C}^d)$.
(ii) $\operatorname{Ker} R_0 = \{f \in \mathscr{O}(\mathbb{C}^d)\colon f|V = 0\}$, $R_0^2 = R_0$.
(iii) *For every* $r \ge 1$,

$$|R_i f|_{rB} \le |f|_{rB}\,, \qquad i = 0\,,\,\ldots\,,\,p\,,$$

where for an entire function

$$f = \sum_{\alpha\in\mathbf{Z}_+^d} f_\alpha \mathbf{Z}^\alpha \quad \text{on } \mathbb{C}^d$$

and $c = (c_1 \cdots c_d)$ *we set*

$$|f|_c \doteq \sum_{\alpha\in\mathbf{Z}_+^d} |f_\alpha| c^\alpha\,.$$

Now we can state the main result of this note:

Theorem 2. *For a closed irreducible subvariety V of $\mathbb{C}^d$ the following conditions are equivalent:*
(i) *V is an algebraic variety.*
(ii) *There exists a linear tame extension operator from $\mathscr{O}(V)$ into $\mathscr{O}(\mathbb{C}^d)$.*

Proof. (i) $\Rightarrow$ (ii) Let V be an algebraic variety in some $\mathbb{C}^d$. Following the notation of Theorem 1, if $f \in \mathscr{O}(V)$, choose by Cartan Theorem B an analytic function $G \in \mathscr{O}(\mathbb{C}^d)$ such that $G|V = f$ and set $\mathscr{E}(f) \doteq R_0(G)$. In view of (ii) of Theorem 1, $\mathscr{E}$ is a well-defined continuous linear extension operator. An important feature of the operators R_i, $0 \le i \le \rho$, of Theorem 1 above is that for any $r > 1$ they extend to be continuous linear operators from $\mathscr{O}(\Delta_{rB})$ into $\mathscr{O}(\Delta_{rB})$ satisfying
(a) $g = R_0(g) + \sum_{i=1}^{\rho} R_i(g)Q_i$ for all $g \in \mathscr{O}(\Delta_{rB})$.
(b) $\operatorname{Ker} R_0 = \{g \in \mathscr{O}(\Delta_{rB})\colon g|V \cap \Delta_{rB} = 0\}$,
where Δ_{rB} is the polydisc around zero with polyradii rB. (See [3, Corollary 3].)

Now for a sufficiently large, fixed r, we will examine the restriction operator from $\mathscr{O}(\Delta_{rB})$ to $\mathscr{O}(\Delta_{rB} \cap V)$. We will use as a fundamental system of norms, the norms

$$\|F\|_{\Delta_{sB}} \doteq \sup_{z \in \Delta_{sB}} |F(z)|, \qquad s < r,$$

for $\mathscr{O}(\Delta_{rB})$ and

$$\|f\|_{\Delta_{sB} \cap V} \doteq \sup_{z \in \Delta_{sB} \cap V} |f(z)|, \qquad s < r, \text{ for } \mathscr{O}(\Delta_{rB} \cap V).$$

Since this restriction operator is a surjection, in view of the open mapping theorem, we can find a $C_0 = C_0(r)$ such that for every $f \in \mathscr{O}(V)$ with $\|f\|_{\Delta_{rB} \cap V} \le 1$, there exists an $F_r \in \mathscr{O}(\Delta_{rB})$ such that $F_r | V \cap \Delta_{rB} = f | V \cap \Delta_{rB}$ and $\|F_r\| \le \Delta_{rB/2} \le C_0$. Taking into account the above-mentioned feature of the operator R_0, we have $\mathscr{E}(f)|\Delta_{rB} = R_0(F_r)$ on Δ_{rB}. Hence for every $r > 1$, sufficiently large, there exists $C_i = C_i(r) > 0$, $i = 0, 1$, such that

$$\begin{aligned} \|\mathscr{E}(f)\|_{\Delta_{rB/3}} = \|R_0(F_r)\|_{\Delta_{rB/3}} &\le |R_0(F_r)|_{rB/3} \le |F_r|_{rB/3} \\ &\le C_1 \|F_r\|_{\Delta_{rB/2}} \le C_0 C_1 \|f\|_{\Delta_{rB} \cap V} \end{aligned} \tag{1}$$

where we used (iii) of Theorem 1. This shows that $\mathscr{E}$ is a linear tame extension operator from $\mathscr{O}(V)$ into $\mathscr{O}(\mathbb{C}^d)$.

(ii) $\Rightarrow$ (i) Suppose that there exists a linear tame extension operator $\mathscr{E}: \mathscr{O}(V) \to \mathscr{O}(\mathbb{C}^d)$ satisfying $\exists A, B$ such that for n sufficiently large, $\exists d_n > 0$;

$$\|\mathscr{E}(f)\|_n \le d_n |f|_{An+B}.$$

Fix a large m_0 so large that $\frac{m_0 - 1}{A} - \frac{B}{A} \gg 1$, and set $K = K_{m_0} \doteq \overline{D}_{m_0} \cap V$. We let $L = \{u \in \mathrm{PSH}(\mathbb{C}^d) : \exists \alpha = \alpha(u)$ such that $u(z) \le \ln(1 + \|z\|) + \alpha$ for $z \in \mathbb{C}^d\}$. Now fix a $u \in L$ such that $u|K \le 0$. At this point we need a lemma which is probably well known but for which we could not find a reference.

Lemma. *Let ρ be a plurisubharmonic function on $\mathbb{C}^d$. Then there exist analytic functions $f_n \in \mathscr{O}(\mathbb{C}^d)$ and positive integers c_n such that*

$$\rho(z) = \overline{\lim_n} \frac{\ln|f_n(z)|}{c_n} \quad \forall z \in \mathbb{C}^d.$$

Proof. First choose a sequence $\{\rho_n\}$ of continuous plurisubharmonic functions such that $\rho_n \downarrow \rho$ pointwise on $\mathbb{C}^d$ (see [5, p. 48]). Let $B_n = \{z \in \mathbb{C}^d : \|z\| \le n\}$, and choose a sequence of positive numbers $\{\varepsilon_n\}_n$ decreasing to zero. For each n there exists a natural number $\rho(n)$, entire functions $\{F_i^n\}_{i=1,\dots,\rho(n)}$ and $\rho(n)$ natural numbers $\{c_i^n\}_{i=1,\dots,\rho(n)}$ such that

$$\rho_n(z) - \varepsilon_n \le \max_{1 \le i \le \rho(n)} \frac{\ln|F_i^n(z)|}{c_i^n} \le \rho_n(z) + \varepsilon_n, \qquad z \in B_n \tag{2}$$

(see [6, p. 55]). We enumerate the doubly indexed sequence $\{F_i^n\}$ as

$$\{F_1', \dots, F_{\rho(1)}', F_1^2, \dots, F_{\rho(2)}^2, \dots, F_1^n, \dots, F_{\rho(n)}^n, \dots\}$$

and denote the resulting sequence by $\{G_\alpha\}$. We also use the same rule to enumerate $\{c_i^n\}$ and denote the resulting sequence by $\{c_\alpha\}$.

Let

$$\gamma_\alpha \doteq \frac{\ln|G_\alpha(z)|}{c_\alpha}, \qquad k_n \doteq \sum_{i=1}^{n-1} \rho(i) + 1, \quad n > 1.$$

Now fix a point $z \in \mathbb{C}^d$ say $z \in B_N$. For $n > N$, our construction yields

$$(3) \qquad \rho(z) - \varepsilon_n \leq \rho_n(z) - \varepsilon_n \leq \sup_{\alpha \geq k_n} \gamma_\alpha(z).$$

On the other hand for $\alpha \geq k_n$, since $\gamma_\alpha(z) = (\ln|F_i^s(z)|)/c_i^s$ for some $s \geq n$ and some i between 1 and $\rho(s)$, we have in view of (2),

$$(4) \qquad \gamma_\alpha(z) \leq \rho_s(z) + \varepsilon_s \leq \rho_n(z) + \varepsilon_n.$$

Let $\varepsilon > 0$ be given. Choose M such that $\varepsilon_M \leq \varepsilon$ and $M > N$. Then (3) and (4) for $n > M$ imply

$$\rho(z) - \varepsilon \leq \inf_n \sup_{\alpha \geq k_n} \gamma_\alpha(z) = \overline{\lim_n} \gamma_n(z) \leq \rho(z) + \varepsilon$$

This proves the lemma.

In view of the above lemma we fix a representation of u, say

$$u(z) = \overline{\lim_n} \frac{\ln|f_n(z)|}{c_n} \quad \forall z \in \mathbb{C}^d.$$

Let s be a large integer, and let $k_s = [[As + B + 1]] + 1$. Since $u(z) \leq \ln(1+e^{k_s})+\alpha$ on D_{k_s}, by Hartog's lemma (see [5, p. 21]), there exists a $C = C(s)$ such that

$$\sup_{z \in \overline{D}_{k_s - 1}} |f_n(z)| \leq Ce^{(\ln(1+e^{k_s})+\alpha+1)c_n} \quad \forall n.$$

Hence there exists a $\widehat{C} = \widehat{C}(s)$ such that

$$(5) \qquad \|\mathscr{E}(f_n)\|_s \leq \widehat{C}e^{(\ln(1+e^{k_s})+\alpha+1)c_n} \quad \forall n.$$

It follows that the sequence of plurisubharmonic functions $c_n^{-1}\ln|\mathscr{E}(f_n)|$ is a locally bounded (from above) family, so the formula

$$\rho(z) = \overline{\lim_{\xi \to z}}\, \overline{\lim_n} \frac{\ln|\mathscr{E}(f_n)(\xi)|}{c_n}$$

defines a plurisubharmonic function ρ on $\mathbb{C}^d$. Moreover, in view of (4) we can find a constant $\beta = \beta(A; B; u) > 0$ such that

$$(6) \qquad \rho(z) \leq A\ln(1 + \|z\|) + \beta \quad \forall z \in \mathbb{C}^d.$$

On the other hand, since $u \leq 0$ on K, in view of Hartog's lemma (this time on the variety V; see [9]) we have

$$|f_n|_{m_0 - 1} \leq Ce^{c_n} \quad \forall n \text{ for some } C = C(m) > 0.$$

Fix an n_0, with $n_0 < (m_0 - 1)/A - B/A$. Then for all n we have

$$\|\mathscr{E}(f_n)\|_{n_0} \leq C|f_n|_{An_0+B} \leq Ce^{c_n} \quad \text{for some } C = C(m).$$

Therefore,

$$(7) \qquad \rho(z) = \overline{\lim_{\xi \to z}}\, \overline{\lim_n} \frac{\ln|\mathscr{E}(f_n)|}{c_n} \leq 1 \quad \text{for } z \in \overline{D}_{n_0}.$$

Now (4) and (5) imply that

$$\rho(z) \leq AV^*(z, \overline{D}_{n_0}) + 1 \quad \text{for all } z \in \mathbb{C}^d$$

where $V^*(z, S)$ for a compact set $S \subseteq \mathbb{C}^d$ is the Siciak extremal function of S defined via

$$V^*(z; S) = \overline{\lim_{\xi \to z}} \sup\{v(\xi); v \leq 0 \text{ on } S, v \in L\}$$

which is known to be a plurisubharmonic function if S is not pluripolar (see [9]). Taking into account the fact that u is dominated by ρ on the variety V we obtain

$$\sup\{u(z); u \in L; u \leq 0 \text{ on } K\} \leq AV^*(z, D_{n_0}) + 1 \quad \forall z \in V.$$

So our assertion now follows from the following result of Sadullaev (see [9]).

Theorem. *A necessary and sufficient condition for an analytic set $A \subseteq \mathbb{C}^d$ to be a piece of an algebraic set is that for some compact set $K \subset A$ the function* $\sup\{u(z); u \in L, u \leq 0$ *on* $K\}$ *belongs to* $L^\infty_{\text{loc}}(A)$.

This finishes the proof of the theorem. □

We remark that the proof of the necessity in the above-mentioned theorem of Sadullaev relies heavily on the geometric criterion for algebraic varieties given in [8].

References

1. A. Aytuna, *Stein spaces M for which $\mathcal{O}(M)$ is isomorphic to a power series space*, Advances in the Theory of Fréchet Spaces, Kluwer Academic Publ., Dordrecht, 1989, pp. 115–154.
2. A. Aytuna, J. Krone, and T. Terzioğlu, *Complemented infinite type power series subspaces of nuclear Fréchet spaces*, Math. Ann. **283** (1989), 193–202.
3. P. B. Djakov and B. S. Mitiagin, *The structure of polynomial ideals in the algebra of entire functions*, Studia Math. **68** (1980), 84–104.
4. R. S. Hamilton, *Nash-Moser inverse function theorem*, Bull. Amer. Math. Soc. (N.S.) **7** (1986), 65–222.
5. L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, Amsterdam, 1973.
6. P. Lelong, *Plurisubharmonic functions and positive differential forms*, Gordon and Breach, New York, 1969.
7. B. S. Mitiagin and G. M. Henkin, *Linear problems of complex analysis*, Russian Math. Surveys **26** (1971), 99–164.
8. W. Rudin, *A geometric criterion for algebraic varieties*, J. Math. Mech. **17** (1967/68), 671–683.
9. A. Sadullaev, *An estimate for polynomials on analytic sets*, Math. USSR-Izv. **20** (1983), 493–502.
10. D. Vogt, *Tame spaces and power series spaces*, Math. Z. **196** (1987), 523–536.
11. ——, *Über die Existenz von Ausdehnungsoperatoren für holomorphe Funktionen auf analytischen Mengen in C^n*, preprint.
12. V. P. Zahariuta, *Isomorphisms of spaces of analytic functions*, Soviet Math. Dokl. **22** (1980), 631–634.

Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

E-mail address: `mataydintrmetu.bitnet`