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# CYCLES IN $C^r$ TWISTS

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ABSTRACT. Let $f_t$, $0 \leq t \leq 1$, be a continuous one-parameter family of $C^r$ diffeomorphisms of the circle obtained by monotonically twisting away from $f_0$. It is well known that for a dense set of parameter values $t$, $f_t$ has a periodic orbit. To what extent does the distribution in the circle of these periodic orbits reflect the degree of differentiability $r$? We show that if $r \geq 6$, $\sup_t \|f_t \circ f_0^{-1}\|_{C^r} < \infty$, and the rotation number of $f_0$ is an irrational $\alpha$ with bounded continued fraction expansion, then periodic orbits corresponding to small values of the parameter $t$ echo the metric structure of $f_0$ in the following sense: If the rotation number of the orbit is a convergent of $\alpha$, then the orbit divides the circle into intervals of nearly equal $\mu$-measure, where $\mu$ is the invariant Borel probability measure of $f_0$. The corresponding result for low differentiability is false.

## 1. INTRODUCTION

Given a $C^r$ orientation-preserving diffeomorphism $f$ of the circle with nontrivial recurrence it is well known that one can obtain a periodic orbit by composing $f$ with an arbitrarily small $C^r$ twist. To what extent does the distribution of the periodic orbits in the circle reflect the degree of differentiability $r$?

The following gives a crude measure of how much a periodic orbit $\mathscr{C} = \{o_i\}_{i=1}^n$ deviates from being a cycle of a rigid rotation. Remove $\mathscr{C}$ from the circle to obtain a finite number of intervals. Define $R(\mathscr{C}) \in (0, 1]$ to be the ratio of the minimal length of such an interval to the maximal length. If $R(\mathscr{C})$ is close to 1, then $\mathscr{C}$ is nearly a cycle of a rigid rotation. More generally, define $R_\mu(\mathscr{C})$ to be the analogous ratio relative to the measure $\mu$, where $\mu$ is a Borel probability measure. Then $R_\mu$ measures the extent to which $\mathscr{C}$ deviates from a cycle dividing the circle into intervals of equal $\mu$-measure.

We say that a $C^0$ one-parameter family of diffeomorphisms of the circle $\{g_t\}$, $0 \leq t \leq 1$, is a *monotone twist* if it lifts to a family $\{G_t\}$ on the real line for which $G_0$ is the identity and for which $t_1 < t_2$ implies $G_{t_1} < G_{t_2}$; $\{f_t\}$ is a *forward twist* of $f$ if it is obtained by composing $f$ by a monotone twist $\{g_t\}$. We suppose that $f = f_0$ leaves invariant the Borel probability measure $\mu$ and consider $R_\mu$ of cycles generated by $f_t$.

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We show that if $f$ has a good rotation number and the $f_t$ are sufficiently differentiable, with $\sup_t \|f_t \circ f^{-1}\|_{C^r}$ bounded (note: $\{f_t\}$ need not be $C^r$ in $t$), then there exists a sequence of periodic orbits of $\{f_t\}$, with selected rotation numbers, for which $R_\mu \to 1$, where $\mu$ is the probability measure left invariant by $f$. Thus these periodic orbits echo the metric structure of $f$.

More precisely,

**Theorem 1.** *Suppose $\{f_t\}$ is a forward twist, $\sup_t \|f_t \circ f_0^{-1}\|_{C^r} < \infty$, $f_t \in C^r(\mathbf{S}^1)$, $r \geq 6$, and $f_0$ has an irrational rotation number $\alpha$ with bounded continued fraction expansion. Let $\{\mathscr{C}_n\}$ be a sequence of cycles generated by $\{f_t\}$ with rotation numbers principal convergents of $\alpha$. Then $R_\mu(\mathscr{C}_n) \to 1$.*

Recall that if $\alpha$ has continued fraction expansion $[a_0; a_1, a_2, \ldots]$ the $n$th principal convergent of $\alpha$ is given by $[a_0; a_1, a_2, \ldots, a_n]$ [4].

On the other hand, in the $C^1$ case, given any rotation number, it is not hard to find sequences of cycles corresponding to principal convergents with ratios that cluster on an entire interval. Such clustering may happen also in the $C^\infty$ case if the terms $\{a_n\}$ of the continued fraction expansion of $\alpha$ grow sufficiently rapidly.

The $C^1$ and $C^r$, $r \geq 6$, cases differ because $C^r$ perturbations have more of a tendency to spread. Another consequence of this spreading is that while in the $C^1$ case it is always possible to produce by a $C^1$-small perturbation of $f$ a cycle which agrees for the most part with a segment of orbit of the original system (whose points need not be well-distributed on $\mathbf{S}^1$), this is not always the case for $C^r$ perturbations, $r \geq 2$. This phenomenon, in higher dimensions, lies at the heart of the well-known closing problem, or whether it is possible to $C^r$-approximate, $r \geq 2$, a flow of diffeomorphism having a recurrent orbit by one having a periodic orbit [3, 2].

Theorem 1 generalizes a result in [1].

## 2. Definitions and notation

To study circle diffeomorphisms with rotation number $\alpha$ it is useful to consider certain Rokhlin towers associated with the rigid rotation $R_\alpha$, as pointed out by Katznelson and Ornstein in [5] (see also [2]).

Let

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_2 + \cdots}}}, \qquad a_0 \in \mathbf{Z}^1 \cup \{0\}, \; a_i \in \mathbf{Z}^+.$$

Define a sequence $\{q_n\}$ as follows:

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}, \qquad n \geq 2.$$

For a point $p \in \mathbf{S}^1$, this is the sequence of times at which its future orbit, $\{\mathscr{R}_\alpha^n(p)\}$, $n \geq 0$, achieves new minimal distances to its original position. Let $d_n$, $n \geq 2$, be the new minimal distance achieved at time $q_n$—that is,

$$d_n = \min(|q_n \alpha \ (\mathrm{mod}\ 1)|, \ 1 - |q_n \alpha \ (\mathrm{mod}\ 1)|).$$

It follows from the definition of the $q_n$ that $d_n = a_{n+2} d_{n+1} + d_{n+2}$.

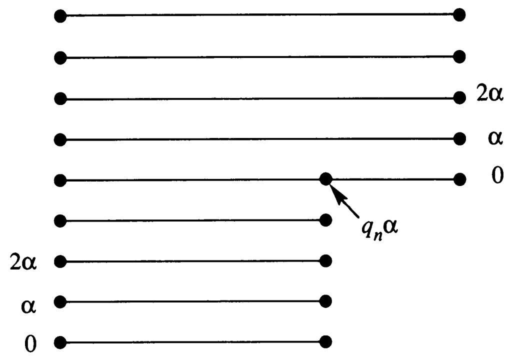


FIGURE 1

If

$$d_n = |q_n\alpha \,(\text{mod } 1)|, \qquad n \geq 2,$$

we say that $q_n$ is a *right return-time*; if

$$d_n = 1 - |q_n\alpha \,(\text{mod } 1)|,$$

we say that $q_n$ is a *left return-time.* It is known that right returns and left returns alternate.

For each $q_n$, the orbit of the point 0 (more properly, $[0] \in \mathbf{R}/\mathbf{Z}$) partitions $\mathbf{S}^1$ into $q_n$ intervals. These may be cut out and rearranged to form a vertical stack, as in Figure 1.

We use "$n\alpha$" elliptically to stand for "$n\alpha\,(\text{mod } 1)$".

We call the above diagram the $q_n$*-tower.* We say that a point has *height* $i$ if it lies in the level whose left endpoint is $i\alpha$.

The action of $R_\alpha$ on the tower is to push points up one level where possible. The uppermost level, assuming $q_n$ is a closest left-return, may be broken up into two pieces: the leftmost, $[(q_n - 1)\alpha, -1\alpha]$, of length $d_n$, the rightmost, $[-1\alpha, (q_{n-1} - 1)\alpha]$, of length $d_{n-1}$. The leftmost interval is mapped to $[q_n\alpha, 0]$, the rightmost, to $[0, q_{n-1}\alpha]$. The case of a closest right-return is treated similarly.

To obtain the $q_{n+1}$th tower from the $q_n$th, we simply follow the orbit of $q_n\alpha$ until the next closest return is achieved and then rearrange the intervals so as to obtain a new tower.

The shape of a $q_n$-tower is roughly determined by the terms of the continued fraction expansion. In particular, the $n$th term gives the number of times the height of the balcony divides that of the lower body of the tower, the $(n+1)$th term, the number of times the width of the balcony divides the width of the main body of the tower.

We will be interested also in certain substacks of intervals, defined for a left return-time, $q_n$, and choice of point $p \in \mathbf{S}^1$:

$$\bigcup_{i=0}^{q_n} \mathscr{R}_\alpha^i[p, \mathscr{R}_\alpha^{-q_n}(p)] = \bigcup_{i=0}^{q_n}[p + i\alpha, p + i\alpha + d_n].$$

We call this stack the $q_n$*-box at* $p$. In Figure 2 we indicate the $q_n$-box at 0.

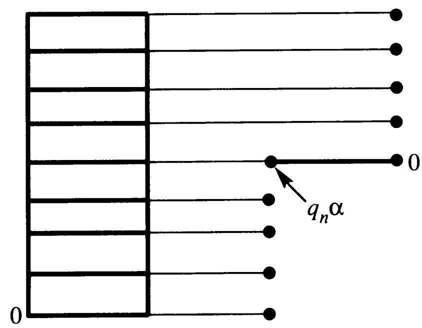


FIGURE 2

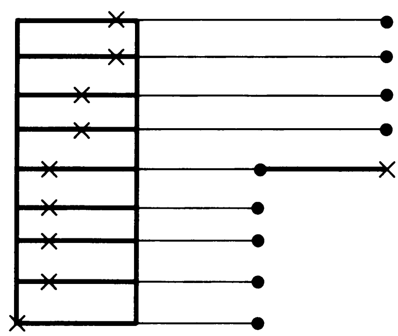

FIGURE 3

## 3. CYCLES AND TRACES

Let $\{f_t\}$ be a forward twist. We first consider the case that $f_0$ is a rigid rotation $R_\alpha$, $\alpha \in \mathbf{R}\backslash\mathbf{Q}$. By simple topological considerations there exists, corresponding to each sufficiently large left return time $q_n$, a parameter value $t_n$ for which $f_{t_n}$ produces a cycle which traverses the $q_n$-box, from corner to corner, meeting the interior of each level at most once. One may check that the rotation number of such a cycle is a principal convergent of the continued fraction expansion of $\alpha$.

To simplify notation we set $f_n = f_{t_n}$. We call the set of orbit points, $\{(f_n)^i(0)\}$, $0 \le i \le q_n$, thought of as elements of the $q_n$-box, trace($f_n$) (see Figure 3).

Consider the sequence of traces generated by $\{f_t\}$. If we renormalize the corresponding $q_n$-boxes (thought of as point sets) to have height 1 and width 1, the shape of traces can be compared.

As long as we stay in the $C^1$-topology, traces are relatively flexible. Let $\gamma$ be any continuous monotonically increasing curve joining the lower-left corner of a square to the upper-right corner. Then no matter what $C^1$ bound is put on the norm of the twist we can find a forward twist whose corresponding traces approximate $\gamma$, up to a rescaling of $q_n$-boxes. If, however, we pass to the $C^r$-topologies, $r \ge 4$, and the terms of the continued fraction expansion of $\alpha$ are bounded, then traces of a forward twist must eventually approach a diagonal.

**Proposition 1.** *Suppose $R_\alpha$, $\alpha \in R\backslash Q$, has rotation number with bounded continued fraction expansion. Let $\{f_n\}$ be any sequence of $C^r$ maps, $r \geq 4$, in a forward twist of $R_\alpha$ such that $f_n$ generates a $q_n$-cycle as above. Assume $\|f_n \circ f_0^{-1}\|_{C^r} < B$. Then, as $n \to \infty$, the associated sequence of renormalized traces approaches a diagonal.*

*Proof.* It will be convenient to regard the $f_n$ as perturbations of $f_0$. We will first show that the point of intersection of trace($f_n$) with a level whose approximate normalized height in the $q_n$-tower is $1/2$ divides the level into pieces of approximately equal size. The idea is that the perturbation, $f_n$, assumed of small $C^r$ size, $r \geq 4$, will have a significant effect at a given level of a $(q_n, 0)$-tower—say, $f_n$ pushes points by $q_n/d_n$—only if it has a significant effect at other levels of the tower. Consequently, the perturbation has a tendency to spread to other levels. Moreover, a push of a significant size induces other pushes of approximately the same size, spread uniformly throughout the $q$-tower. This means that the distance moved by the orbit of 0 along the lower levels roughly balances that moved along the upper levels.

We will use $h_n$ to denote the distance a point is moved by $f_n \circ f_0^{-1}$—that is, $h_n = |F_n - T|$, where $T(x) = x + \alpha$ and $F_n$ is a lift of $f_n$ to $\mathbf{R}$ such that $0 < |F_n(x) - T(x)| < 1$ for all $x \in \mathbf{R}$. By hypothesis $\|h_n\|_{C^r} < 1$.

The range of values that $h_n$ may take is constrained by the fact that it generates a $q_n$-cycle, as follows. Let $\|h_n\|_{C^0} = a$, $a \in \mathbf{R}^+$. Then it follows from a nice estimate due to Kolmogorov [6] that the fastest rate of decrease from the value $a$ that is compatible with the bound on the $C^r$-norm is $\frac{K_{r-1}}{K_r^{(r+1)/r}} a^{(r-1)/r} B^{1/r}$, where $K_i$ is a Favard constant,

$$K_i = \begin{cases} \left(\frac{4}{\pi}\left(1 - \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} - \frac{1}{7^{i+1}} + \cdots\right)\right)^{-1} & \text{if } i \text{ is even}, \\ \left(\frac{4}{\pi}\left(1 + \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} + \frac{1}{7^{i+1}} + \cdots\right)\right)^{-1} & \text{if } i \text{ is odd}. \end{cases}$$

(I thank M. Stessin for pointing out this estimate to me.) So $h_n(x) > a/2$ on an interval of length greater than $C_r a^{1/r}$, where $C_r$ is a constant depending only on $r$ and $B$. We follow the convention that $C_r$ denotes a constant that depends only on $r$ and $B$, but we will allow this constant to vary from line to line, as convenient. It follows from Kolmogorov's estimate that the number of levels of the $q_n$-tower on which $h_n(x) > a/2$ is bounded below by $C_r a^{1/r}/(2d_{n-1})$. (Recall that $d_{n-1} + d_n < 2n_{d-1}$ is the length of the widest level of $q_n$-tower.) Hence the orbit of the point 0, as it moves up the tower, will travel a total horizontal distance bounded below by:

$$C_r a^{1/r}/d_{n-1} \times a/2 > C_r a^{1/r} \frac{1}{(a_{n+1}+1)d_n} \times a/2 > C_r a^{1+1/r}(a_{n+1}d_n)^{-1}.$$

Since $f_n$ generates a $q_n$-cycle, this distance in turn is bounded above by $d_n$, so

$$\begin{aligned} C_r a^{1+1/r} d_n^{-1} a_{n+1}^{-1} &< d_n, \\ C_r a^{1+1/r} &< d_n^2 a_{n+1}, \\ a &< C_r d_n^{2r/(r+1)} a_{n+1}^{r/(r+1)}. \end{aligned}$$

We record also the following obvious lower bound:

$$a = \|h_n\|_{C^0} \geq \frac{d_n}{q_n}.$$

By computing sums of lengths of levels in a $q_n$-tower we have

$$q_n d_{n-1} + q_{n-1} d_n = 1\,,$$
$$(q_n a_{n+1} + q_{n-1}) d_n + q_n d_{n+1} = 1\,.$$

Thus $a_{n+1} q_n d_n = 1$ and the lower bound noted above may be replaced by $a_{n+1} d_n^2$.

In order to understand how the values taken on by $h_n$ are distributed in the $q_n$-tower, we must first understand how the intervals of $S^1$ are distributed in the $q_n$-tower. Recall the levels of the $q_n$-tower (or, equivalently, the first $q_n$ points of the orbit of 0) induce a natural decomposition of $S^1$ into intervals. Given $\varepsilon \in \mathbf{R}^+$, we will show that it is possible to select $l(\varepsilon) \in Z^+$ such that for sufficiently large $q_n$ any $l$ levels of the $q_n$-tower constituting an interval of $S^1$ are distributed approximately evenly about a "middle" level—that is, letting $J_n$ be the middle level, the difference between the number of levels that lie below $J_n$ and the number that lie above $J_n$ is less than $\varepsilon l$.

It will be convenient to assume that towers have been renormalized so that their left boundary sits in an interval $[0, 1]$, the first level being at 0 and the top level at $q_{n-1}/q_n$. We define a map, $s_n$, on the left boundary of each renormalized tower for a left return as follows: $s(y) =$ the left endpoint of the interval immediately to the right of that corresponding to $y$. Recall the identification of the left and right boundaries in a tower. This map agrees with the rational rotation $y \to y + (q_{n-1}/q_n) (\mathrm{mod}\ 1)$. It is not difficult to check that for each $n \in \mathbf{Z}^+$ the terms of the continued fraction expansion of $q_{n-1}/q_n$ are uniformly bounded by some $K < \infty$, the bound being the same as that on the terms of the continued fraction expansion of the original rotation number $\alpha$. It follows that for $q_n$ large a sufficiently long segment of an orbit of $s$—the bound on the length being independent of the choice of orbit or of $q_n$—visits any given interval of measure $1/2$ approximately half the time. In particular, this holds for the upper and lower halves of the domain of $s$, giving us a way of choosing $l$. We choose $l$ so that in any segment of orbit of $s$ of length $l$ the number of extra visits to the upper or lower half-interval is less than $\varepsilon l$. It is not hard to see that a similar choice can be made for intervals of length $1/N$ rather than $1/2$.

Fix $\varepsilon$, $l(\varepsilon)$, and $q_n$, where $q_n$ is assumed large relative to $l(\varepsilon)$. Consider the decomposition of $S^1$ induced by levels of the $q_n$-tower. Divide $S^1$ into groups, each of which consists of $l$ levels whose union forms an interval of $S^1$. There may be a few levels left over—at most $l-1$—which will be taken to constitute a last group. We now argue that the amount of progress made by the orbit of 0 along each such group is either negligible or is split roughly evenly between upper and lower levels.

Consider first the group, if there is one, that consists of fewer than $l$ levels. The total horizontal distance moved by 0 along these levels is bounded above by $(l-1) \times C_r d_n^{1+(r-1)/(r+1)} a_{n+1} \le (l-1) \times C_r d_n d_n^{(r-1)/(r+1)}$. If $n$ is sufficiently large, $l$ will be an arbitrarily small percentage of $q_{n-1}$, which quantity is eventually insignificant relative to $d_n$.

Next consider those groups on which the push is small relative to $d_n^2$, say, less than $\varepsilon d_n^2$. Then the total horizontal distance traveled on the union of these is at most $q_n \varepsilon d_n^2 < \varepsilon d_n$. For $n$ large, this again is negligible relative to $d_n$.

This leaves groups $I_1, I_2, \ldots, I_m$ on which $h_n$ takes on a value greater than or equal to $\varepsilon d_n^2$. We may estimate the extent to which the value of $h_n$ can change over such a group, $I_i$, as follows. The maximum value of the derivative of $h_n$, estimated in terms of the maximum value of $h_n$, is less than

$$C_r(d_n^{1+(r-1)/(r+1)} a_{n+1}^{r/(r+1)})^{(r-1)/r} = C_r d_n^{2(r-1)/(r+1)}.$$

Let $x_i \in I_i$ be a point at which the maximum value over the $l$ intervals is achieved, and let $x_i+\delta$ be any other point in the $l$-sequence of intervals. Then

$$\begin{aligned}\frac{h_n(x_i+\delta)}{h_n(x_i)} &\leq \frac{h_n(x_i)+(C_r d_n^{2(r-1)/(r+1)} \times \delta)}{h_n(x_i)} \\ &\leq 1+\frac{C_r d_n^{2(r-1)/(r+1)} \times l d_n}{\varepsilon d_n^2} \\ &= 1+\frac{1}{\varepsilon} C_r d_n^{(r-3)/(r+1)}.\end{aligned}$$

Note that this quantity tends to 1 as $n$ increases provided $r \geq 4$.

It follows that for $n$ sufficiently large, $h_n$ is approximately constant along $I_i$. We may assume $n$ has been chosen so that

$$1-\varepsilon \leq \frac{h_n(x+\delta)}{h_n(x)} \leq 1+\varepsilon,$$

where $x$, $x+\delta$ are any two points in $I_i$.

Given that $h_n$ is nearly constant along $I_i$ and that the intervals in each group are divided approximately evenly between the upper and lower portions of the tower, the orbit of 0 should, intuitively, have traveled approximately half the total distance, or $d_n/2$, by the time it reaches the "middle" level. More rigorously, separate the levels of each $I_i$ "upper" and "lower" levels. The choice of $l$ allows us to establish a one-to-one correspondence between the upper and lower levels with at most $\varepsilon l$ odd levels left out of the correspondence—the *extra* intervals. Let $x_i \in I_i$. Then the ratio of the distance moved by the orbit along the extra intervals of $I_i$ to the total distance moved along $I_i$ will be bounded above by

$$\frac{\varepsilon l \times (1+\varepsilon) h_n(x_i)}{(1-\varepsilon) l \times (1-\varepsilon) h_n(x_i)} = \frac{\varepsilon(1+\varepsilon)}{(1-\varepsilon)^2}.$$

Given that this estimate holds over all $I_i$, the ratio of the distance traveled by the orbit of 0 along all the extra intervals of all the $I_i$ to the total distance traveled over all the $I_i$ itself satisfies this estimate. Hence, if $\varepsilon$ is chosen very small, the distance traveled over the extra intervals is a negligible percentage of the total distance traveled.

Finally we come to the intervals we have decided do matter, namely, those that enter into the correspondence between upper and lower intervals for the $I_j$. Although we now have an equal number of levels above $J_n$ and below $J_n$, the distances traveled over each set of levels may still differ. Let $l_i \leq l$ be the number of intervals in $I_i$ entering into the correspondence. There is a maximum possible extra push in favor of the upper or lower levels of $l_i \times 2\varepsilon h_n(x_j)$, $x_j \in I_j$. This is a small percentage of the total push contributed by these intervals. For supposing $\varepsilon < 1/2$, this quantity is less than

$$\frac{(l_i/2) \times 2\varepsilon h_n(x_j)}{(l_i/2) \times (1-\varepsilon) h_n(x_j)} = \frac{2\varepsilon}{1-\varepsilon} < 4\varepsilon.$$

The estimate is independent of the choice of $I_j$, so that the total possible skewing on the $I_j$ is bounded above by $4\varepsilon d_n$.

Bringing together these various estimates, we may conclude that by the time we reach the "middle" level, $J_n$, the orbit of the point 0 will have moved a distance of roughly $d_n/2$. In other words, the sequence of points $\{\mathrm{Trace}(f_n) \cap J_n\}$ converges to the midpoint of $J_n$, as claimed.

Consider now, for $q_n$ large, $\{\mathrm{trace}(f_n) \cap (\bigcup_i L_i)\}$, where $L_i$ is the $i$th of $M-1$ levels that divide the $q_n$-tower into $M$ substacks of approximately equal height. Then by a variant of the argument just given these sets converge to points $p_i \in L_i$, $1 \leq i \leq M-1$, $p_i$ positioned at approximately $i/M$ of the way from the left endpoint of $L_i$. Since $M$ was arbitrary and since the traces are monotonic, it follows that $\mathrm{trace}(f_n)$ approaches a diagonal. □

Theorem 1 now follows as an easy corollary when $f_0$ is a rigid rotation. Given a cycle $\mathscr{C}$ corresponding to a convergent of $\alpha$, $R(\mathscr{C})$ is simply the ratio of the minimum distance, measured along $\mathbf{S}^1$, between two points of the associated trace, to the maximum such distance. Since traces become nearly linear as the convergents approach $\alpha$, we must have $R(\mathscr{C}) \to 1$. We do not know whether a similar result holds for $r = 2$ or 3.

If $f_0$ is not a rigid rotation but of class at least $C^6$, then given our assumption on the rotation number it follows from results of Herman and Yoccoz [9] that it is $C^4$ conjugate to a rigid rotation. Applying the above proposition, we obtain cycles, corresponding to principal convergents of the rotation number of $f_0$, for parameter values tending to 0 for which $R_\mu \to 1$, where $\mu$ is the measure left invariant by $f_0$. This completes the argument.

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