PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

AXIOMATIZATION AND UNDECIDABILITY RESULTS FOR METRIZABLE BETWEENNESS RELATIONS

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(Communicated by Andreas R. Blass)

ABSTRACT. Let d be a metric on a nonempty set A. The ternary betweenness relation T_d induced by d on A is defined by

$$T_d(x, y, z) \Leftrightarrow d(x, y) + d(y, z) = d(x, z)$$

for $x, y, z \in A$. Allowing the range of d to vary over some "reasonable" ordered additive algebraic structures (not just the real numbers), we will prove that the class $\mathscr{M}$ of all metrizable ternary structures, i.e., the class of all structures (A, T_d), where d is some metric on A, is an elementary class which can be axiomatized by a set of universal Horn sentences. Further, using an algorithm of linear programming, we will show that the first-order theory of $\mathscr{M}$ is recursively axiomatizable and its universal part is decidable. On the other hand, the theory of $\mathscr{M}$ is not finitely axiomatizable and the theory of finite members of $\mathscr{M}$ is hereditarily undecidable.

In his fundamental work *Grundlagen der Geometrie* [H 1899] David Hilbert introduced the ternary relation of betweenness which he used in formulating the second group of his axioms, namely the axioms of order. (In fact "y is between x and z" in Hilbert's formulation corresponds to "y lies between x and z and $x \neq y \neq z$" in ours.) Later on Alfred Tarski [T 1959] used the betweenness relation and the quaternary equidistance or congruence relation (i.e., "the distance of x, y and z, u is the same") to obtain a complete and decidable axiomatization of elementary geometry. Since the 1920s the notion of betweenness became of interest in its own right, as it naturally occurred in diverse mathematical structures, like linearly and partially ordered sets and lattices (see [B 1967], [Bl 1953]), metric spaces (see [M 1928], [Bl 1953]), linear spaces over ordered fields and normed linear spaces (see [Sm 1943], [Bl 1953]), etc. However, because of its intuitively clear geometrical meaning acquired in some "nice", though by far not all, metric spaces, the metric betweenness, defined as above, still has kept its central position. So far the paper [Mo 1977] by M. Moszyńska seems to be the only contribution to the study of the first-order theory of betweenness and equidistance relations in metric spaces.

Received by the editors March 23, 1993 and, in revised form, May 25, 1993.

1991 *Mathematics Subject Classification*. Primary 03B30, 51F99, 51K05; Secondary 03B25, 03C52, 08A02.

Key words and phrases. Betweenness relation, geometry, metric, ordered group, ordered field, first-order theory, elementary class, axiomatization, universal Horn sentence, decidable, hereditarily undecidable, interpretation, semantic embedding, linear programming, graph.

It can easily be seen that in any metric space (A, d) the relation $T = T_d$ satisfies the following five axioms:

(B0) $T(x, y, x) \Rightarrow x \equiv y$,

(B1) $T(x, x, y)$,

(B2) $T(x, y, z) \Rightarrow T(z, y, x)$,

(B3) $T(x, y, z) \ \& \ T(x, z, u) \Rightarrow T(x, y, u)$,

(B4) $T(x, y, z) \ \& \ T(x, z, u) \Rightarrow T(y, z, u)$.

On the other hand, the "prolongation" axiom

$$T(x, y, z) \ \& \ T(y, z, u) \ \& \ y \not\equiv z \Rightarrow T(x, y, u)$$

need not hold in general.

A first-order structure (A, T) with a single ternary relation T will be called a *ternary structure;* if it additionally satisfies the axioms (B0) – (B4), it will be called a *betweenness space.*

Every metric space (A, d) gives rise to a betweenness space (A, T_d). However, as is well known, the satisfaction of the axioms (B0) – (B4) is not sufficient for the metrizability of (A, T). Though the metric betweenness (in real-valued metric spaces) was characterized by A. Wald [Wd 1931] in terms of conditions (B0) – (B4) (in a slightly different formulation), topological closedness of the segments $(xy)_T = \{z;\ T(x, z, y)\}$, invariance under the equidistance relation, and possibility of an extension to larger metric spaces with points lying properly between any given pair (see also [Bl 1953]), so far no set of axioms in the first-order language with equality $\equiv$ and a single ternary predicate T has been known whose models were exactly the metrizable ternary structures (A, T). Even the existence of such a set, i.e., of the axiomatizability of the class of all metrizable betweenness spaces, was not clear.

One reason is quite trivial. Insofar as the term "metric" means exclusively a real-valued metric, the class of all metrizable betweenness spaces cannot be an elementary one. Indeed, take the ternary relation

$$T(x, y, z) \Leftrightarrow |x - y| + |y - z| = |x - z|$$

on $\mathbb{R}$, induced by the metric $|x - y|$. Let ${}^*\mathbb{R}$ denote some elementary extension of the ordered field $\mathbb{R}$ with cardinality greater than that of $\mathbb{R}$, guaranteed by the upward Löwenheim-Skolem-Tarski theorem, and let ${}^*T$ be the ternary relation on ${}^*\mathbb{R}$ corresponding to T (i.e., defined by the same formula). Obviously, $({}^*\mathbb{R}, {}^*T)$ is an elementary extension of $(\mathbb{R}, T)$. Then, as easily seen, the metrizability of $({}^*\mathbb{R}, {}^*T)$ by a metric $d : {}^*\mathbb{R} \times {}^*\mathbb{R} \to \mathbb{R}$ would yield an injective mapping $a \mapsto d(a, 0)$ from the set of all nonnegative elements in ${}^*\mathbb{R}$ into $\mathbb{R}$ — a contradiction. Consequently, being not closed under elementary extensions, the ternary structures metrizable by real-valued metrics do not form an elementary class. The same argument applies when $\mathbb{R}$ is replaced by any fixed ordered field or even by a more general algebraic structure.

Thus one cannot expect any positive results in the direction of axiomatizability unless one allows some metrics more general than the real-valued ones. We will show several ways of doing this, differing in generality at first glance, but still giving the same elementary class of metrizable ternary structures. Then we will prove that all the axioms have rather a simple prescribed form, in fact they

can be chosen to be basic Horn formulas. By the way, notice that the axioms (B0) – (B4) are basic Horn formulas as well. Further, a simple procedure will be described, reducing the question whether a disjunction of atomic formulas and negations of atomic formulas holds in all metrizable betweenness spaces or not to the solvability problem of a system of linear equations and inequalities, which can be decided using linear programming. In particular, as all the basic Horn formulas are of that form, this ensures the recursive axiomatizability of the first-order theory of the class of all metrizable betweenness spaces. In general, it yields a positive solution of the decision problem for universal formulas in that theory. On the other hand, not only is the first-order theory of the class of all metrizable betweenness spaces not finitely axiomatizable, but so far we are not able to give any lucid scheme of axioms for that class, and believe we have some good reasons to doubt whether such an axiom scheme even exists. Finally, using the fact that the first-order theory of the class of all finite graphs is hereditarily undecidable, we will establish the same result for the class of all finite metrizable betweenness spaces by the method of semantic embedding (interpretation). Thus in particular, the theory of metrizable betweenness spaces is undecidable.

We will use freely the usual terminology and notation common in model theory as well as some results belonging to model-theoretical folklore. The standard reference is the monograph [C–K 1990]. The needed facts concerning (un)decidability of first-order theories can be found either in [Br–Sn 1981] or in [E 1980]. For (partially) ordered monoids, groups, and fields the reader can consult either [F 1963] or [Kp 1984].

1. G-VALUED METRIC SPACES

Let $\mathbf{K} = (K, +, 0, <)$ be a partially ordered (additively written) commutative monoid satisfying the cancellation law (the last condition is equivalent to the preservation of the strict partial order $<$ by the addition $+$). A $\mathbf{K}$-valued metric space is a set A equipped with a function $d : A \times A \to K$, called metric, subject to the usual axioms:

$$\begin{aligned} &d(x, y) \geq 0, \\ &d(x, y) = 0 \Leftrightarrow x = y, \\ &d(x, y) = d(y, x), \\ &d(x, y) + d(y, z) \geq d(x, z). \end{aligned}$$

A ternary structure (A, T) will be called *metrizable* if $T = T_d$ for some metric $d : A \times A \to K$ where $\mathbf{K}$ is a partially ordered commutative monoid with cancellation and T_d is defined as in the Abstract. Obviously, every metrizable ternary structure satisfies the axioms (B0) – (B4); in other words, it is a betweenness space.

At a glance, all one needs for the above four conditions on a metric to make sense is $\mathbf{K}$ to be only a partially ordered monoid. However, in order not to escape too far from the real-valued metric spaces as well as to be able to prove some reasonable results, the commutativity and cancellation assumptions seem unavoidable. If $\mathbf{K}$ additionally satisfies the implication $v + v > 0 \Rightarrow v > 0$ (which is not always the case), then even the first condition on the metric d can be omitted, as it follows from the remaining ones. On the other hand, it would

be convenient and desirable if one could work with metrics taking values in still more "tame" structures without loss of generality. The first candidates are $\mathbf{G}$-valued metric spaces where $\mathbf{G} = (G, +, 0, <)$ is an ordered (i.e., linearly or totally ordered) Abelian group.

Proposition 1. *Let (A, T) be a metrizable betweenness space. Then there is an ordered Abelian group $\mathbf{G}$ and a metric $d : A \times A \to G$ such that $T = T_d$.*

Proof. Let $\mathbf{K}$ be a partially ordered commutative monoid with cancellation and $h : A \times A \to K$ be a metric inducing T. Then $(K, +, 0)$ can be embedded into the Abelian group $(D, +, 0)$ of the differences of the elements from K and the partial order from K can be extended to D in the obvious way, making $\mathbf{D} = (D, +, 0, <)$ a partially ordered Abelian group containing $\mathbf{K}$ as a substructure. Thus h can be viewed as a $\mathbf{D}$-valued metric. Let us denote by P the positive cone and by Θ the subgroup of torsion elements in $\mathbf{D}$. As $P \cap \Theta = \{0\}$, Θ is a convex subgroup of $\mathbf{D}$ and the quotient group $\mathbf{D}/\Theta$ becomes partially ordered in the natural way. Let g denote the canonical projection $\mathbf{D} \to \mathbf{D}/\Theta$. Then g is an order-preserving group homomorphism and $h(x, y) \in P$ for all $x, y \in A$. Hence $d(x, y) \notin \Theta$ for $x \neq y$ and the map $d = g \circ h : A \times A \to D/\Theta$ is a $\mathbf{D}/\Theta$-valued metric. The inclusion $T = T_h \subseteq T_d$ follows from the group homomorphy of g; as $h(x, y) + h(y, z) - h(x, z) \geq 0$ for all $x, y, z \in A$, the reversed inclusion is a consequence of the equality $P \cap \Theta = \{0\}$, again. As the partially ordered group $\mathbf{D}/\Theta$ is torsion-free, its partial order $<$ can be extended to a total order $\prec$ (see [F 1963], Chapter III, §4). Hence (A, T) is metrized by the metric d with values in the ordered Abelian group $\mathbf{G} = (D/\Theta, +, 0, \prec)$.

If $\mathbf{F} = (F, +, \cdot, 0, <)$ is an ordered ring, then an $\mathbf{F}$-valued metric space is simply an $\mathbf{F}^+$-valued metric space, where $\mathbf{F}^+ = (F, +, 0, <)$ denotes the ordered additive group of $\mathbf{F}$.

The following proposition may be known, though we did not succeed to find it anywhere in the literature. That is why we state it here and give the complete proof.

Proposition 2. *Every ordered Abelian group can be embedded into the ordered additive group of an ordered field.*

Proof. Obviously, it suffices to consider the case when $\mathbf{G}$ is an ordered Abelian group with more than one element. It is known (see [E 1980], Chapter 3, §6) that every ordered Abelian group $\mathbf{G}$ can be embedded into an ordered divisible Abelian group $\widehat{\mathbf{G}}$ and the first-order theory of nontrivial ordered divisible Abelian groups is complete. Thus given any ordered field $\mathbf{F}$, $\widehat{\mathbf{G}}$ and the ordered additive group $\mathbf{F}^+$ of $\mathbf{F}$ are elementarily equivalent as $\mathbf{F}^+$ obviously is nontrivial and divisible. Consequently, $\widehat{\mathbf{G}}$ can be embedded into some ultrapower of $\mathbf{F}^+$, which, of course, is the ordered additive group of the corresponding ultrapower of the ordered field $\mathbf{F}$.

2. Metrizable betweenness spaces

Proposition 1 from the previous section justifies the following restriction of our original definition which will be used in the rest of the paper. A ternary structure (A, T) will be called metrizable if $T = T_d$ for some $\mathbf{G}$-valued metric d on A where $\mathbf{G}$ is an ordered Abelian group. As follows from Proposition

2, for our purpose it would even be sufficient to deal with $\mathbf{F}$-valued metric spaces where $\mathbf{F} = (F, +, \cdot, 0, <)$ ranges over ordered fields. Moreover, as every ordered field can be embedded into a real-closed one and the first-order theory of real-closed ordered fields is complete, each $\mathbf{G}$-valued metric space can be considered as a $^*\mathbb{R}$-valued metric space for some elementary extension $^*\mathbb{R}$ of the ordered field $\mathbb{R}$.

Theorem. *Let $\mathscr{M}$ denote the class of all metrizable betweenness spaces and* $\mathrm{Th}\mathscr{M}$ *be its first-order theory. Then the following conditions hold*:

(i) *$\mathscr{M}$ is an elementary class.*
(ii) $\mathrm{Th}\mathscr{M}$ *has a set of universal Horn axioms.*
(iii) $\mathrm{Th}\mathscr{M}$ *is recursively axiomatizable.*
(iv) *The universal part of* $\mathrm{Th}\mathscr{M}$ *is decidable.*
(v) $\mathrm{Th}\mathscr{M}$ *is not finitely axiomatizable.*
(vi) *The class of all finite members of $\mathscr{M}$ is hereditarily undecidable; hence* $\mathrm{Th}\mathscr{M}$ *is undecidable.*

Proof. (i) and (ii): $\mathscr{M}$ obviously is closed under taking isomorphic images and substructures. In particular, it is closed under elementary substructures. Thus to prove that $\mathscr{M}$ is a universal elementary class, it suffices to show that it is closed under the ultraproduct construction. This can easily be seen as $\mathscr{M}$ is the class of all reducts (A, T) of two-sorted structures of the form $(A, G, T, +, 0, <, d)$ such that $(G, +, 0, <)$ is an ordered Abelian group, $d : A^2 \to G$ is a metric, and $T = T_d$, which obviously form an elementary class.

A universal class is a Horn one iff it is closed with respect to direct products of couples of its members (see [C–K 1990], Chapter 6, §2). To show it for $\mathscr{M}$, consider two structures $(A_i, T_i) \in \mathscr{M}$ $(i = 1, 2)$ and denote by (A, T) their direct product. Let d_i be a metric on A_i, with values in an ordered Abelian group $\mathbf{G}_i$, inducing T_i. Let us denote by $\mathbf{G} = (G_1 \times G_2, +, 0, <)$ the ordered Abelian group obtained by endowing the direct product of the group reducts of $\mathbf{G}_i$'s with the lexicographical order. The verification that the ternary structure (A, T) is metrized by the $\mathbf{G}$-valued metric d given by

$$d((a_1, a_2), (b_1, b_2)) = (d_1(a_1, b_1), d_2(a_2, b_2)),$$

for $a_i, b_i \in A_i$, is a matter of routine now.

(iii) and (iv): A formula φ will be called a basic formula if it is of the form $\theta_1 \vee \cdots \vee \theta_m$ where each θ_l is an atomic formula or negation of an atomic formula. We will say that a formula ψ occurs in such a basic formula φ if ψ is among the formulas $\theta_1, \ldots, \theta_m$. According to (i) and (ii) it suffices to describe an algorithm deciding for each basic formula φ in the language of $\mathscr{M}$ whether $\mathscr{M} \models \varphi$ or not. Then (the universal closures of) the basic Horn formulas φ (i.e., those containing at most one atomic θ_l) which are true in $\mathscr{M}$ will form a recursive set of axioms for $\mathrm{Th}\mathscr{M}$. Furthermore, the decidability question for any universal formula (after putting it into prenex form with matrix consisting of a conjunction of basic formulas) can be reduced to the same question for its maximal basic subformulas.

Now, let φ be a basic formula of the form $\theta_1 \vee \cdots \vee \theta_m$ where each θ_l is an atomic formula or negation of an atomic formula with variables included in the list $x_1, \ldots, x_n$. Let us denote by Σ_φ the system of linear equations and

inequalities in the unknowns d_{ij}, $i, j = 1, \dots, n$ (d_{ij} stands for the distance $d(x_i, x_j)$), constructed as follows:

$$\begin{aligned}
d_{ij} &= 0 \text{ if } i = j \text{ or the formula } x_i \not\equiv x_j \text{ occurs in } \varphi,\\
d_{ij} - d_{ji} &= 0 \text{ for all distinct } i, j,\\
d_{ij} + d_{jk} - d_{ik} &= 0 \text{ if } \neg T(x_i, x_j, x_k) \text{ occurs in } \varphi,\\
d_{ij} &> 0 \text{ if } x_i \equiv x_j \text{ occurs in } \varphi,\\
d_{ij} &\geq 0 \text{ if } i \neq j \text{ and neither } x_i \equiv x_j \text{ nor } x_i \not\equiv x_j \text{ occurs in } \varphi,\\
d_{ij} + d_{jk} - d_{ik} &> 0 \text{ if } T(x_i, x_j, x_k) \text{ occurs in } \varphi,\\
d_{ij} + d_{jk} - d_{ik} &\geq 0 \text{ if neither } T(x_i, x_j, x_k) \text{ nor } \neg T(x_i, x_j, x_k) \text{ occurs in } \varphi.
\end{aligned}$$

Of course, Σ_φ in general will contain many redundant equations and inequalities, which can be omitted in concrete situations. Now, it can easily be seen that the condition

(0) $\mathscr{M} \models \varphi$

is equivalent to any of the following ones:

(1) Σ_φ has no solution in any ordered Abelian group $\mathbf{G}$;
(2) Σ_φ has no solution in any ordered divisible Abelian group $\mathbf{G}$;
(3) Σ_φ has no solution in the ordered field $\mathbb{R}$ of reals;
(4) Σ_φ has no solution in the ordered field $\mathbb{Q}$ of rationals;
(5) Σ_φ has no solution in the ordered group $\mathbb{Z}$ of integers.

Indeed, (0) $\Leftrightarrow$ (1) as well as the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. To show (5) $\Rightarrow$ (1), assume that Σ_φ has a solution in some ordered Abelian group $\mathbf{G}$. Without loss of generality we can assume that $\mathbf{G}$ is divisible. Then $\mathbf{G}$ can be embedded into some ultrapower of $(\mathbb{Q}, +, 0, <)$, hence Σ_φ has a solution in that ultrapower and consequently in $\mathbb{Q}$. By multiplying all the numbers $d_{ij} \in \mathbb{Q}$ by the least common multiple of their denominators one obtains a solution of Σ_φ in $\mathbb{Z}$. Obviously, any of the problems (3), (4), and (5) can be decided using a standard algorithm of linear or integer programming (see, e.g., [S 1986]).

Also, note that in the case of φ being a basic Horn formula, the system Σ_φ contains at most one strict inequality, which, in view either of (3) or (4), can be replaced by the equality of the left side to 1. More precisely, the new system $\Sigma_\varphi^\star$ thus obtained has a solution in $\mathbb{R}$ or $\mathbb{Q}$ iff the original system Σ_φ has.

(v): According to a result of Vaught [Vt 1954] (see also [C–K 1990], Chapter 5, §2, Exercise 2), it suffices to show that for each sufficiently large $n \in \mathbb{N}$ there is a ternary structure $(A, T) \notin \mathscr{M}$ such that each n-element substructure of it is in $\mathscr{M}$.

First of all note that given any metric d on the four-element set $\{a_0, a_1, b_0, b_1\}$, the conditions $T_d(a_0, b_0, b_1)$ and $T_d(b_0, a_0, a_1)$ imply $d(a_0, b_0) \leq d(a_1, b_1)$. Indeed, in such a case

$$2d(a_0, b_0) = d(a_0, b_1) - d(b_0, b_1) + d(b_0, a_1) - d(a_0, a_1) \leq 2d(a_1, b_1).$$

Now, let A_n denote for each $n > 1$ the $(2n+2)$-element set $\{a_0, b_0, \dots, a_n, b_n\}$, and T_n be the ternary relation on A_n consisting of all the triples of the form (x, x, y) $(x, y \in A_n)$, (a_i, b_i, b_{i+1}), (b_i, a_i, a_{i+1}) $(0 \leq i \leq n)$ and the reversed ones where $+$ denotes addition mod $n+1$. The structure (A_3, T_3)

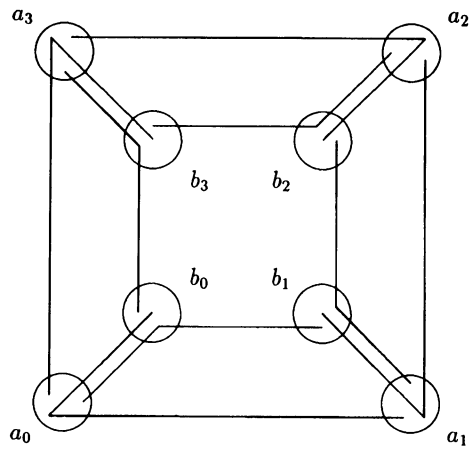

DIAGRAM 1

is drawn in Diagram 1 (for three distinct vertices x, y, z there is a (maybe broken) line form x to z passing through y if and only if $T(x, y, z)$ holds).

It can easily be verified that any of the ternary structures (A_n, T_n) satisfies the axioms (B0) – (B4), hence it is a betweenness space. On the other hand, none of the betweenness spaces (A_n, T_n) with $n > 1$ is metrizable, since for every metric d on A_n inducing T_n the equalities $d(a_0, b_0) = d(a_1, b_1) = \cdots = d(a_n, b_n)$ necessarily hold. But then $T_n(a_i, a_{i+1}, b_{i+1})$ and $T_n(b_i, b_{i+1}, a_{i+1})$ for each $i \leq n$ ($+$ denotes again addition mod $n+1$). The proof of (v) will be complete once we show that omitting the triples (a_n, b_n, b_0), (b_n, a_n, a_0) and the reversed ones from T_n, the ternary structure (A_n, T_n°) thus obtained already is metrizable. Then by a symmetry argument one can show that for any subset $B \subseteq A_n$ with at most n elements (then $a_i, b_i \notin B$ for some $i \leq n$) the ternary structure $(B, T_n \cap B^3)$ is metrizable. Let p be any real number such that $0 < p < 1/(n+1)$. We leave it to the reader to verify that the metric d_n given for $0 \leq i, j \leq n$ by

$$d_n(a_i, a_j) = d_n(b_i, b_j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases}$$

$$d_n(a_i, b_j) = d_n(b_j, a_i) = \begin{cases} (i+1)p & \text{if } i = j, \\ (i+1)p + 1 & \text{if } |i-j| = 1, \\ 1 & \text{if } |i-j| > 1 \end{cases}$$

induces the relation T_n° on A_n.

(vi): We will prove the last statement by constructing a semantic embedding of the class of all finite graphs into the class of all finite members of $\mathscr{M}$. By the theory of graphs we understand the first-order theory in the language with equality and a single binary predicate symbol E, with two axioms expressing the irreflexivity and symmetry of the relation corresponding to E. Intuitively, in a graph (V, E) two vertices $u, v \in V$ are connected by an edge iff $(u, v) \in E$. It is known that the theory of the class of all finite graphs is hereditarily undecidable (see, e.g., [Br–Sn 1981], Chapter V, §5, or [E 1980], Chapter 5, §1). Thus any class into which the class of finite graphs can be semantically embedded has itself a hereditarily undecidable first-order theory.

We will construct an interpretation of the theory of graphs in the theory of betweenness spaces defining a unary predicate $W(x)$, with meaning "x is a vertex", and a binary predicate $H(x, y)$, with meaning "x and y are connected by an edge", by the following formulas in the language of ternary structures:

$$W(x) \Leftrightarrow (\forall y, z)(T(y, x, z) \Rightarrow x \equiv y \vee x \equiv z),$$
$$H(x, y) \Leftrightarrow W(x) \ \& \ W(y) \ \& \ (\exists z)(x \not\equiv z \not\equiv y \ \& \ T(x, z, y)).$$

It is clear that for each (finite) betweenness space (A, T) the pair $(W^{A,T}, H^{A,T})$, where

$$W^{A,T} = \{a \in A;\ (A, T) \models W(a)\},$$
$$H^{A,T} = \{(a, b) \in A^2;\ (A, T) \models H(a, b)\},$$

is a (finite) graph. What remains is to show that every finite graph (V, E) is of the form $(W^{A,T}, H^{A,T})$ for some finite *metrizable* betweenness space (A, T).

Let (V, E) be a finite graph. As the set V is finite, it can obviously be embedded into the Euclidean plane $\mathbb{R}^2$ via a map $f : V \to \mathbb{R}^2$ in such a way that for no triple of distinct vertices $u, v, w \in V$ the points $f(u)$, $f(v)$, $f(w)$ are collinear. (In fact it is enough for f to be a one-to-one mapping from V into the parabola $y = x^2$.) Using f we will identify each vertex $v \in V$ with its image $f(v)$. The finiteness of V also implies that for any pair of vertices $(u, v) \in E$ one can choose a point a_{uv} on the segment uv in such a way that $u \neq a_{uv} = a_{vu} \neq v$ and putting

$$A = V \cup \{a_{uv};\ u, v \in V \ \& \ (u, v) \in E\},$$

there will be no three-element collinear subsets of A except those of the form $\{u, a_{uv}, v\}$. We make A a metric space (A, d) under the restriction of the Euclidean metric in $\mathbb{R}^2$ to A. Let $T = T_d$ and (A, T) be the corresponding finite metrizable betweenness space. From our construction it is clear that $(V, E) = (W^{A,T}, H^{A,T})$.

Now, the proof of the theorem is complete.

Let us conclude with some remarks.

Any basic formula φ can be considered as a "transitivity" in the sense of [Pt–Sm 1942]. However, in this paper only transitivities with at most five variables are considered. On the other hand, as shown in the proof of (v), any axiomatization of the class $\mathscr{M}$ of all metrizable betweenness spaces has to use "transitivities" with arbitrarily large number of variables. Maybe it is worthwhile to mention here that, as one can easily check, all the betweenness spaces (i.e., all ternary structures satisfying the axioms (B0) – (B4)) with at most five elements are metrizable. The smallest nonmetrizable betweenness spaces have six elements — the structure (A_2, T_2) from part (v) of the proof of our theorem is an example.

Every basic Horn formula φ of the form $\theta_1 \& \cdots \& \theta_m \Rightarrow \theta_0$ where $\theta_0, \theta_1, \ldots, \theta_m$ are atomic formulas which is true in $\mathscr{M}$ can be viewed as describing a "forbidden pattern" for $\mathscr{M}$. More precisely, if $x_1, \ldots, x_n$ are all the variables in φ, then $\mathscr{M} \models \varphi$ means that the "pattern"

$$\theta_1(x_1, \ldots, x_n) \ \& \ \cdots \ \& \ \theta_m(x_1, \ldots, x_n) \ \& \ \neg\theta_0(x_1, \ldots, x_n)$$

cannot occur in any member of $\mathscr{M}$. For example, any of the nonmetrizable betweenness spaces (A_n, T_n) $(n > 1)$ is an example of a "forbidden pattern" (cf.

Diagram 1). The corresponding basic Horn formula α_n forbidding (A_n, T_n) reads as follows:

$$\begin{aligned} T(x_0, y_0, y_1) \ \& \ T(y_0, x_0, x_1) \ \& \ \cdots \ \& \ T(x_{n-1}, y_{n-1}, y_n) \\ \& \ T(y_{n-1}, x_{n-1}, x_n) \ \& \ T(x_n, y_n, y_0) \\ \& \ T(y_n, x_n, x_0) \Rightarrow T(x_0, x_1, y_1). \end{aligned}$$

This could suggest the idea to axiomatize $\mathscr{M}$ by the axioms (B0) – (B4) and the scheme $\{\alpha_n;\ 1 < n \in \mathbb{N}\}$ or, if necessary, to complete it by some additional axioms and axiom schemes. Unfortunately, there are many sequences of nonmetrizable betweenness spaces producing new forbidding axiom schemes, similarly as the structures (A_n, T_n) produce the axioms α_n. Moreover, they can be constructed in such a way that all the structures from any one of the corresponding sequences satisfy all the forbidding formulas produced by the remaining sequences. As the structures from different sequences and their various fragments can be combined and put together in diverse manners, there seems to be little hope for finding an axiomatization of the class $\mathscr{M}$ consisting of a reasonable number of readable axiom schemes. On the other hand, this could be the explanation for the length of the period for which the axiomatization problem for metrizable betweenness spaces resisted solution.

Finally, let us turn back to the class $\mathscr{M}_0$ of all betweenness spaces metrizable by real-valued metrics. As already pointed out, $\mathscr{M}_0$ is not an elementary class. Nevertheless, to ask for its first-order theory or, equivalently, for the smallest elementary class containing $\mathscr{M}_0$, still makes sense. As the finite members of $\mathscr{M}$ obviously belong to $\mathscr{M}_0$, both the classes $\mathscr{M}$ and $\mathscr{M}_0$ satisfy the same universal formulas and, by (vi), the theory $\mathrm{Th}\,\mathscr{M}_0$ is hereditarily undecidable. However, the question whether the smallest elementary class containing $\mathscr{M}_0$ coincides with $\mathscr{M}$ or is a proper subclass of it remains open.

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