

## THE ARC LENGTH OF THE LEMNISCATE $\{|p(z)| = 1\}$

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**ABSTRACT.** We show that the length of the set

$$\left\{ z \in \mathbb{C} : \left| \prod_{i=1}^n (z - \alpha_i) \right| = 1 \right\}$$

is at most  $8\pi en$ . This gives the correct rate of growth in a long-standing open problem of Erdős, Herzog, and Piranian and improves the previous bound of  $74n^2$  due to Pommerenke.

In 1958 Erdős, Herzog, and Piranian [2] raised a number of problems concerned with the lemniscate

$$E_n := E_n(p) := \{z \in \mathbb{C} : |p(z)| = 1\}$$

where  $p$  is a monic polynomial of degree  $n$ , so

$$p(z) := \prod_{i=1}^n (z - \alpha_i), \quad \alpha_i \in \mathbb{C}.$$

One in particular, Problem 12, conjectures that the maximum length of  $E_n$  is achieved for  $p(z) := z^n - 1$  (which is of length  $2n + O(1)$ ). The best partial to date is due to Pommerenke [7], who shows that the maximum length is at most  $74n^2$ . This problem has been reposed by Erdős several times, including recently at a Budapest meeting honouring his eightieth birthday. (See also [3].) It now carries with it a cash prize from Erdős of \$250.

This note derives an upper bound of  $8\pi en$ , which gives the correct rate of growth.

**Theorem.** Let  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ . Then the length of

$$E_n := \left\{ z \in \mathbb{C} : \left| \prod_{i=1}^n (z - \alpha_i) \right| = 1 \right\}$$

is at most  $8\pi en$  ( $\leq 69n$ ).

The proof relies on two classical theorems. One due to Cartan, and one due to Poincaré.

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**Cartan's Lemma** [1, p. 174]. If  $p(z) := \prod_{i=1}^n (z - \alpha_i)$ , then the inequality

$$|p(z)| > 1$$

holds outside at most  $n$  circular discs, the sum of whose radii is at most  $2e$ .

**Poincaré's Formula** [8, 9]. Let  $\Gamma$  be a rectifiable curve contained in  $\mathbb{S}$  (the Riemann sphere). Let  $v(\Gamma, x)$  denote the number of times that a great circle consisting of points equidistant from the antipodes  $\pm x$  intersects  $\Gamma$ . (If this is infinite, set  $v(\Gamma, x) = 0$ .) Then the length of  $\Gamma$ ,  $L_{\mathbb{S}}(\Gamma)$ , is given by

$$L_{\mathbb{S}}(\Gamma) = \frac{1}{4} \int_{\mathbb{S}} v(\Gamma, x) dx$$

where  $dx$  is the area measure on  $\mathbb{S}$ .

We need the following corollary of this result.

**Corollary.** Suppose  $\Gamma$  is an algebraic curve in  $\mathbb{R}^2$  of degree at most  $N$  and  $D$  is a disc of radius  $R$ . Then the length of  $\Gamma \cap D$  is at most  $2\pi RN$ .

*Proof.* By an affine scaling it suffices to prove this for  $D$  a disc of radius 1 about the origin. Now any conic intersects  $\Gamma$  in at most  $2N$  points by Bezout's theorem. It follows that the projection of  $\Gamma$  in the Riemann sphere is intersected by any great circle in  $\mathbb{S}$  in at most  $2N$  points. Thus the length of the projection of  $\Gamma$  in  $S$  is at most  $2\pi N$  by Poincaré's formula. Since the projection back to the unit disc does not increase arclength, the result is proved.  $\square$

We can now prove the theorem.

*Proof of Theorem.* Fix  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ . By Cartan's Lemma there exist circles  $D_1, \dots, D_m$  with radius  $r_1, \dots, r_m$  so that

$$E_n \subset \bigcup_{i=1}^m D_i \quad \text{and} \quad \sum_{i=1}^m r_i \leq 2e.$$

Observe that  $E_n$  is an algebraic curve in  $\mathbb{R}^2$  of degree at most  $2n$  in  $x$  and  $y$  where  $z = x + iy$ . So by the corollary each disc  $D_i$  contains a portion of  $E_n$  of length at most  $4\pi r_i n$ . Summing over  $i$  we deduce that the length of  $E_n$  is at most  $8\pi en$ .  $\square$

The constant 2 in the corollary can be removed with some effort, so a sharpening to  $4\pi en$  is possible. The constant  $e$  in Cartan's Lemma is probably unnecessary, but this is open. Even with these improvements we would only get a bound of  $4\pi n$  which still is not sharp. Indeed, it seems likely that this type of method is too blunt to yield an exact result.

There are a number of interesting related results. See for example Pommerenke [4–7]. In Pommerenke [6] it is shown that if the roots in the theorem are all real, then the length is at most  $4\pi$ .

In Pommerenke [5] it is shown that if the set  $E_n$  is connected, then the length is at least  $2\pi$ , with equality only for  $z^n$ . When  $E_n$  is connected one can find a disc of radius 2 that contains it [5]. So in this case the length of  $E_n$  is at most  $4\pi n$ .

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