APPLICATION OF THE OPERATOR PHASE SHIFT 
IN THE \( L \)-PROBLEM OF MOMENTS

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(Communicated by Palle E. T. Jorgensen)

Abstract. This note studies more deeply the results obtained in an earlier pa-
112 (1991)). It gives a similar condition for the solvability of the \( L \)-problem 
of moments, using the operator phase shift. Based on this, it underlines some 
of the aspects of the operator phase shift used in the \( L \)-problem of moments.

0. Introduction

The \( L \)-problem of moments consists of characterizing the moment sequence

\[
A_n = \int_{\mathbb{R}} t^n B(t) \, dt, \quad n \in \mathbb{N},
\]

of a measurable operator-valued function \( 0 \leq B(\lambda) \leq L \).

In the scalar case, this problem was formulated and completely solved by 
Achiezer and Krein in 1930. The problem can be formulated in the same man-
ner for operator-valued functions. The solvability of the operator \( L \)-problem of 
moments can be linked with the phase shift introduced in the theory of operator 
perturbation.

The phase shift is a completely unitary invariant that characterizes a pertur-
bation pair of two operators.

This invariant was introduced by Carey in [2]. In this article, it was proved 
that the principal function \( \phi(z) = I + K(A-z)^{-1}K^* \), \( \text{Im } z \neq 0 \), of the operator 
pair \( \{A, K\} \) admits an exponential representation

\[
\Phi(z) = \exp \left( \int_{\mathbb{R}} \frac{B(\lambda)}{\lambda - z} \, d\lambda \right)
\]

where \( B(\lambda) \) is a summable function taking values in the positive cone of the 
unit ball of the bounded operators.

This function was called the phase shift. Based on this exponential represen-
tation, we gave in [7] a solvability condition for the \( L \)-problem of moments.
In this note, we take again the idea of linking the exponential representation of the phase shift with the \( L \)-problem of moments and give another solvability condition.

Further, in the same article, Carey gives a characterization of the spectral type of the operator that was perturbed and also a characterization of the support of the phase shift.

We relate these concepts to the \( L \)-problem of moments. Hence, we give a condition in terms of moments which implies that the spectral measures of the appearing operators are absolutely continuous; also we give conditions of the moments which imply that the phase shift has a given support.

\section{1}

In this note, one of our aims is to prove the equivalent conditions 1° \( \equiv \) 4° \( \equiv 5° \) from the following theorem (we mention that the equivalent conditions 1° \( \equiv 2° \equiv 3° \) were proved in [7]).

\textbf{Theorem 1.0.} The following assertions are equivalent:

1°. The sequence \((A_n)_{n=0}^{\infty}\) represents the successive moments of a summable operator-valued function, \( 0 \leq B(\lambda) \leq L \).

2°. There is another operator sequence \((A'_n)_{n=0}^{\infty}\) for which we have the equality

\[
\exp \left( -L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1}
\]

with both operatorial matrices \((A'_{m+n})_{m,n=0}^{\infty}\) and \((-A'_{m+k+2} + C_1 A'_{m+k})_{m,k=0}^{\infty}\) nonnegatively defined for \( C_1 \) a positive constant.

3°. There is a spectral measure \( \sigma : \text{Bor}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) for which

\[
I + \int_{\mathbb{R}} \frac{d\sigma(t)}{t-z} = \exp \left( -L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right).
\]

4°. There is another operator sequence \((A''_n)_{n=0}^{\infty}\) for which we have the relation:

\[
\exp \left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I + \sum_{m=0}^{\infty} A''_m z^{-m-1}
\]

with both operatorial matrices \((A''_{n+m})_{n,m=0}^{\infty}\) and \((-A''_{m+k+2} + C_2 A''_{m+k})_{m,k=0}^{\infty}\) nonnegatively defined for \( C_2 \) a positive constant.

5°. There is a spectral operator measure \( \sigma' : \text{Bor}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) for which:

\[
I - \int_{\mathbb{R}} \frac{d\sigma'(t)}{t-z} = \exp \left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right).
\]

\textbf{Proof.} The equivalent conditions 1° \( \equiv \) 2° \( \equiv 3° \) were proved in [7]. We shall prove in this note that conditions 1°, 4°, and 5° are equivalent. We assume first that 1° is true; that is, \((A_n)_{n=0}^{\infty}\) represents the sequence of successive moments of a summable operator-valued function \( B : \mathbb{R} \to \mathcal{L}(\mathcal{H}) \), \( 0 \leq B(\lambda) \leq L \), i.e., \( A_n = \int_{\mathbb{R}} t^n B(t) \, dt \), \( n \in \mathbb{N} \).

From the boundedness of the function \( B \), it implies \( 0 \leq L^{-1} B(t) \leq I \), so we have \( L^{-1} A_n = \int_{\mathbb{R}} t^n L^{-1} B(t) \, dt \).
The first step is a reduction of the power series of the moments to a Cauchy integral formula. Computing the sum for \( N \)-indices we obtain

\[
\sum_{n=0}^{\infty} L^{-1} A_n z^{-n-1} = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{t^n}{z^{n+1}} L^{-1} B(t) \, dt = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}} L^{-1} B(t) \, dt = \int_{\mathbb{R}} L^{-1} B(t) \frac{1}{z-t} \, dt = -\int_{\mathbb{R}} \frac{L^{-1} B(t)}{t-z} \, dt
\]

for \( z \) sufficiently large.

We shall note that \( B'(t) = L^{-1} B(t) \). For this new function, we have \( 0 \leq B'(t) \leq I \).

Carey's result shows that a summable operator function can be the phase shift of a perturbation pair; i.e., if \( \tilde{B} \) is a \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \)-valued operator function with \( 0 \leq \tilde{B} \leq I \), then

\[
I + K(A-z)^{-1} K^* = \exp \left( \int_{\mathbb{R}} \frac{\tilde{B}(t)}{t-z} \, dt \right), \quad \text{Im} z \neq 0.
\]

Instead of \( z \), we take \(-z\) in this formula, \( \text{Im} -z \neq 0 \). With this change, the equality becomes

\[
I + K(A-(-z))^{-1} K^* = \exp \left( \int_{\mathbb{R}} \frac{\tilde{B}(t)}{t-(-z)} \, dt \right)
\]

equivalent with

\[
I - K(A' - z)^{-1} K^* = \exp \left( -\int_{\mathbb{R}} \frac{B'(-\lambda)}{\lambda-z} \, d\lambda \right)
\]

where we have noted that \( A' = -A \), \( \lambda = -t \), and \( B'(-\lambda) = \tilde{B}(-t) \); because \( 0 \leq \tilde{B} \leq I \), that implies \( 0 \leq B' \leq I \) and \( 0 \leq B \leq L \). The operator \( A' \) is also selfadjoint, and the function \( B' \) is also a summable operator function which takes values in the cone of the unit ball of \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \) operators having

\[
\text{supp } B' = -\text{supp } \tilde{B}.
\]

For this representation we have

\[
\exp \left( -\int_{\mathbb{R}} \frac{B'(-\lambda)}{\lambda-z} \, d\lambda \right) = \exp \left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I - K(A'-z)^{-1} K^* = \tilde{\phi}(z).
\]

This exponential representation of the phase shift will help us construct the \((a_n^n)_{n=0}^{\infty}\) sequence

\[
\tilde{\phi}(z) = I - K(A'-z)^{-1} K^* = I + K \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}} K^*.
\]

We identify \( A'' = KA'n K^* \).

From both representations of the principal function \( \tilde{\phi}(t) \) we obtain

\[
\exp \left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I + \sum_{m=0}^{\infty} A_m' z^{-m-1},
\]

the required equality.
We shall prove that the obtained matrix \((A''_{n+m})_{n,m=0}^{\infty}\) is nonnegatively defined (i.e., \(\sum_{m,k} A''_{m+k} x_m x_k \geq 0\) for every \((x_m)_{m=0}^{\infty}\) sequence with finite support).

With the definition of \(A''_m\), the inequality becomes

\[
\sum_{k,m} \langle KA''_{m+k} K^* x_k, x_m \rangle = \sum_{k,m} \langle A''_{k} K^* x_k, A''_{m} K^* x_m \rangle = \left| \sum_{m=0}^{\infty} A''_{m} K^* x_m \right|^2 \geq 0.
\]

We shall prove in the second turn that there is a constant \(C_2 > 0\), so that the matrix \((-A''_{m+k+2} + C_2 A''_{m+k})_{n,m=0}^{\infty}\) is nonnegatively defined.

With the definition of \(A''_m\) this condition becomes

\[
\sum_{m,k} \langle KA''_{m+k+2} K^* x_m, x_k \rangle \leq \sum_{m,k} C_2 \langle KA''_{m+k} K^* x_m, x_k \rangle.
\]

Indeed,

\[
\left| \sum_{m=0}^{\infty} A''_{m} K^* x_m \right|^2 \leq \left| \sum_{m=0}^{\infty} A''_{k} K^* x_m \right|^2,
\]

an inequality that is true for \(\sqrt{C_2} = \|A''\| > 0\).

We shall prove that \(4^\circ\) implies \(5^\circ\). Suppose that there is an operator sequence \((A'_{n})_{n=0}^{\infty}\) for which the matrices \((A''_{n+m})_{n,m=0}^{\infty}\) and \((-A''_{n+k+2} + C_2 A''_{n+k})_{n,m=0}^{\infty}\) are nonnegatively defined and for which we have

\[
\exp \left( - \sum_{n=0}^{\infty} A''_{n} z^{-n-1} \right) = I + \sum_{n=0}^{\infty} A''_{n} z^{-n-1}.
\]

We shall prove the existence of a spectral positive measure with the required property. For this, we consider the operator sequence \((A'_{n})_{n=0}^{\infty}\) to be doubly indexed. With the assumption, \((A''_{m})_{m=0}^{\infty}\) can be represented as an operator-valued, positively defined function

\[
A'' : N \times N \to \mathcal{L} (\mathcal{H}), \quad A''(m, n) = A''_{m+n}.
\]

The classical Kolmogorov theorem gives a decomposition for positively defined kernels:

Let \(K : I \times I \to \mathcal{L} (\mathcal{H})\) be a positively defined operator-valued function (i.e., \(\sum_{i,j} \langle K(i, j) x_i, x_j \rangle \geq 0\) for every family \((x_i)_{i=0}^{\infty}\) with finite support). Then \(K(i, j)\) admits a decomposition of the form \(K(i, j) = h_i^* h_j\) with \(h_i \in \mathcal{L} (\mathcal{H})\).

Thus, \(A''_{m+n}\) can be represented as \(A''_{m+n} = K''_{n+m} K''_{m}\). From the nonnegativity condition together with Kolmogorov's decomposition, it follows that we can find a constant \(C_2 > 0\) so that \((-K''_{n+1} K''_{m+1} + C_2 K''_{n} K''_{m})_{n,m=0}^{\infty}\) is nonnegatively defined. According to this, for \((x_k)_{k=0}^{\infty}\) an arbitrary family of vectors of finite support, we have

\[
C_2 \sum_{m,k} \langle K''_{m} K''_{k} x_k, x_m \rangle \geq \sum_{m,k} \langle K''_{m+1} K''_{k+1} x_k, x_m \rangle,
\]

an inequality which becomes

\[
\sqrt{C_2} \left| \sum_{k=0}^{\infty} K''_{k} x_k \right| \geq \left| \sum_{k=0}^{\infty} K''_{k+1} x_k \right|.
\]
We take by definition \( A'(\sum_{k=0}^{\infty} K_kx_k) = \sum_{k=0}^{\infty} K_{k+1}x_k \).

Since the \( K_n \) are linear, so is \( A' \) and, from our previous remark, \( A' \) is continuous. Taking \( x_0 = (1, 0, \ldots) \), \( x_i = 0 \) for \( i > 1 \), we obtain \( A'K_0 = K_1 \); and using the induction method for a suitable choice of \( (x_n)_{n=0}^{\infty} \), we obtain \( K_n = A''K_0 \). We shall prove next that the obtained \( A' \) operator is selfadjoint. For \( x \) in a dense subset of \( \mathcal{H} \), we can find \( (x_k)_{k=0}^{\infty} \) such that \( x = \sum_{k=0}^{\infty} K_kx_k \).

In this case, \( \langle A'x, x \rangle = \sum_{k=0}^{\infty} K_{k+1}x_k \sum_{k=0}^{\infty} K_kx_k \in \mathbb{R} \) because \( K_nK_{n+1} \) are positively defined. Thus, \( \langle A'x, x \rangle \in \mathbb{R} \) for any \( x \in \mathcal{H} \) and so \( A' \) is a selfadjoint operator. Because \( A' \) is selfadjoint, it admits a representations of the form \( A' = \int_\mathbb{R} t dE(t) \) where \( \{E_\lambda\}_\lambda \) is its spectral resolution of the unity. Then we have

\[
A''_m = \int_\mathbb{R} t^m d(K_0^*E(t)K_0) \quad \text{for} \quad m \geq 1 \quad \text{and} \quad A''_0 = K_0^*K_0 = \int d\sigma'(t),
\]

where we have noted the measure \( \sigma' : \text{Bor}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) defined by \( \sigma'(\Delta) = K_0^*E(\Delta)K_0 \). Considering this measure, the relation from \( 4^\circ \) becomes

\[
\exp\left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I - \int_\mathbb{R} d\sigma'(t) \frac{t}{t-z},
\]

the required equality.

We shall prove now that \( 5^\circ \) implies \( 1^\circ \). Let \( \sigma' : \text{Bor}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) be a positive measure, satisfying the equality

\[
I - \int_\mathbb{R} d\sigma'(t) \frac{t}{t-z} = \exp\left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right).
\]

With the help of this positive measure, we shall construct the function \( F(t) = \sigma'(\infty, t) \), the function that will be shown to be bounded and nondecreasing (i.e., for \( t_1 < t_2 \), \( F(t_1) \leq F(t_2) \)); that is, \( \sigma'(\infty, t_2) \leq \sigma'(\infty, t_1) + \sigma'[t_1, t_2] \).

It remains to show that \( \sigma'[t_1, t_2] \geq 0 \); it means that \( \langle \sigma'[t_1, t_2], x, x \rangle \geq 0 \), an equality that is true because \( \sigma' \) proceeds from a spectral measure.

We are now able to apply Naimark's dilation theorem [6, Appendix, Theorem 1].

From this theorem, there is a bounded linear mapping \( K \) from an auxiliary Hilbert space \( \mathcal{H} \) into \( \mathcal{H} \) and a resolution of the unity \( \{E_\lambda\}_\lambda \) in \( \mathcal{H} \) such that \( F(\lambda) = KE_\lambda K^* \).

Let \( A' \) designate the selfadjoint operator whose resolution of the unity is \( \{E_\lambda\} \). We evidently have

\[
(1.3) \quad I - \int_\mathbb{R} d\sigma'(t) \frac{t}{t-z} = I - K(A' - z)^{-1}K^* = \exp\left( L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = \tilde{\phi}(z).
\]

From now on, we apply again Carey's result together with the remark in (1.1).

Suppose \( \tilde{\phi}(z) \) is the determining function of a perturbation pair \( \{A', K\} \) on a Hilbert space \( \mathcal{H} \), \( A' \) a selfadjoint operator acting on \( \mathcal{H} \), and \( K : \mathcal{H} \to \mathcal{H} \). Then there is a summable function \( B'(\lambda) \) with values in the set of the positive operators of the unit ball of \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \) such that

\[
\tilde{\phi}(z) = \exp\left( - \int_\mathbb{R} B'(t) \frac{dt}{t-z} \right) = I - K(A' - z)^{-1}K^*.
\]
From both representations (1.1) and (1.3) we have

\[ L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} = \sum_{n=0}^{\infty} \left( \int_{\mathbb{R}} t^n B'(t) \, dt z^{-n-1} \right), \]

that is, \( A_n = \int_{\mathbb{R}} L t^n B'(t) \, dt, \quad n \in \mathbb{N}. \)

By noting \( B(t) = L B'(t), \) we obtain

\[ A_n = \int_{\mathbb{R}} t^n B(t) \, dt, \quad n \in \mathbb{N}. \]

This assertion completes the proof of the theorem.

2

Our next result is based on the previous theorem. We give a characterization of the support of the function \( B(\lambda) \) and a necessary and sufficient condition for \( A \) and \( A' \) operators to be absolutely continuous. These properties are expressed in terms of moments.

**Proposition.** The solution of the moment problem \( B(\lambda) \) has the support included in \([m, M]\) if and only if the matrices \((A'_{n+k+1} - mA'_{n+k})_{n,k=0}^{\infty}\) and \((MA''_{n+k} - A''_{n+k+1})_{n,k=0}^{\infty}\) are nonnegatively defined. ((\(A'_n\))_{n=0}^{\infty} and (\(A''_n\))_{n=0}^{\infty} are the sequences obtained from the previous theorem.)

**Remark.** The consideration about the support is true in the case that we have only the lower or the upper bound of it; in that case it remains only one nonnegativity condition for the two matrices.

**Proof.** If we have \( \text{supp} B(\lambda) \subseteq [m, +\infty), \) from Carey's Proposition 4.1 [2], \( \text{supp} \sigma = \sigma(A) \subseteq \text{supp} B(\lambda) \subseteq [m, +\infty). \) We shall prove the nonnegativeness of the matrix \((A'_{n+k+1} - mA'_{n+k})_{n,k=0}^{\infty}\); that is,

\[ \sum_{n,k=0}^{\infty} \langle (A'_{n+k+1} - mA'_{n+k}) x_n, x_k \rangle \]

\[ = \sum_{n,k=0}^{\infty} \left\langle \int_{\mathbb{R}} (t^{n+k+1} - mt^{n+k}) d\sigma(t) x_n, x_k \right\rangle \]

\[ = \int_{\mathbb{R}} (t-m) d \left( \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t^n x_n, \sum_{k=0}^{\infty} t^k x_k \right) \geq 0 \quad (\text{for } t = t \text{ on } \mathbb{R}). \]

Conversely. We have \( \sum_{n,k=0}^{\infty} \langle (A'_{n+k+1} - mA'_{n+k}) x_n, x_k \rangle \geq 0 \) for every \((x_n)_{n=0}^{\infty} \in \mathcal{H}\) with finite support; that is,

\[ \int_{\mathbb{R}} (t-m) d \left( \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} t^n x_n, \sum_{n=0}^{\infty} t^n x_n \right) \geq 0. \]

This inequality implies that \( t \geq m, \) equivalent with \( \text{supp} \sigma \subseteq [m, +\infty). \)

From Carey's Proposition 4.1 [2], \( \text{supp} \sigma \subseteq \text{supp} B(\lambda), \) an inclusion equivalent with \( \text{supp} B(\lambda) \subseteq [m, +\infty) \).

With the construction in Theorem 1.0, remark (1.2), \( B'(t) = \tilde{B}(-t); \) if \( \text{supp} \tilde{B}(t) \subseteq [-M, +\infty), \) then \( \text{supp} B'(t) \subseteq (-\infty, M] \) and by Carey's result

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supp $\sigma' \subseteq \text{supp } B'(\lambda) = \text{supp } B(\lambda) \subseteq [-\infty, M]$. We shall prove the nonnegativity of the matrix $(MA''_{n+k} - A''_{n+k+1})_{n,k=0}^\infty$; that is,

$$\sum_{n,k=0}^\infty \langle (MA''_{n+k} - A''_{n+k+1})x_n, x_k \rangle$$

$$= \sum_{n,k=0}^\infty \left( \int_\mathbb{R} (Mt^{n+k} - t^{n+k+1}) d\sigma'(t)x_n, x_k \right)$$

$$= \int_\mathbb{R} (M - t) d\sigma' \left( \sum_{n=0}^\infty t^n x_n, \sum_{k=0}^\infty t^k x_k \right) \geq 0.$$

Conversely, if $(MA''_{n+k} - A''_{n+k+1})_{n,k=0}^\infty$ is positively defined, it implies that for every $(x_n) \in \mathcal{H}$ with finite support,

$$\int_\mathbb{R} (M - t) d\sigma' \left( \sum_{n=0}^\infty t^n x_n, \sum_{k=0}^\infty t^k x_k \right) \geq 0,$$

that is, $t \leq M$, equivalent with supp $\sigma' \subseteq (-\infty, M]$ and so supp $B'(t) \subseteq (-\infty, M]$; supp $B(t) = \text{supp } B(t) \subseteq (-\infty, M]$.

We consider now the case in which $B$ has a compact support; we define in this case the function $I: \mathbb{R} \rightarrow B(\mathcal{H}, \mathcal{H})$ by

$$I(t) = \begin{cases} LI(t) & \text{for } t \in \text{supp } B, \\ 0 & \text{for } t \in C \text{ supp } B. \end{cases}$$

This function is summable, with values in the set of positive operators of the $L$-ball of $B(\mathcal{H}, \mathcal{H})$. In this case, $L^{-1}I$ can be the phase shift of a perturbation pair of two operators. We shall define the $L$-moments of this function, i.e.,

$$b_n = \int_\mathbb{R} t^n I(t) \, dt, \quad n \in \mathbb{N}$$

(the integral exists because $I$ has a compact support). In [2] the upper right numerical oscillation of $B(\lambda)$ on $(a, b)$ was introduced by setting:

$$W(B, a, b) = \sup \{ ([B(\lambda'') - B(\lambda')]y, y), a < \lambda' < \lambda'' < b, \|y\| = 1 \}$$

and

$$W(B, \lambda) = \lim_{\lambda \uparrow \lambda_0, \lambda \downarrow \lambda} W(B, a, b).$$

With these definitions, Carey proved in [2, Theorem 4.6] the following:

Let $\{V, K\}$ be a perturbation pair with phase $B(\lambda)$. The operator $V$ is absolutely continuous on the open set $E_B = \{ \lambda | W(\lambda, B) < 1 \}$.

In this paper, with the help of the numerical oscillation of $B(\lambda)$, the phase shift used in Theorem 1.0, we introduce the functions

$$S_{\lambda_0}(t) = \begin{cases} W(\lambda_0, B)I(t) & \text{for } t \in \text{supp } B, \\ 0 & \text{for } t \in C \text{ supp } B \end{cases}$$

for every $\lambda_0 \in \mathbb{R}$.

These functions are summable, taking values in the set of positive operators of the $L$-ball of $B(\mathcal{H}, \mathcal{H})$ and having compact support.
For them, there exist the moments:

\[ a^{(\lambda_0)} = \int_{\mathbb{R}} t^n s_\lambda(t) \, dt \quad \text{for } n \in \mathbb{N}. \]

By using the introduced notions, we have the following proposition:

**Proposition.** The operators \( A \) and \( A' \) from Theorem 1.0 are absolutely continuous on the set \( M \) if and only if \( \sum_{n+k=0}^{\infty} (b_{n+k} - a_{n+k}^{(\lambda_0)}) w_n w_k > 0 \) for every \( \lambda_0 \in M \cap \text{supp } B \) and \( \sum_{n+k=0}^{\infty} (b_{n+k} - a_{n+k}^{(\lambda_0)}) w_n w_k = 0 \) for \( \lambda_0 \in C \text{ supp } B \cap M \).

The proof of this Proposition is evident; it is based only on Carey's result [2, Theorem 4.6].

**ACKNOWLEDGMENT**

The author thanks Palle Jorgensen for his useful advice.

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