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APPLICATION OF THE OPERATOR PHASE SHIFT IN THE L-PROBLEM OF MOMENTS

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ABSTRACT. This note studies more deeply the results obtained in an earlier paper of the author (*An operator-valued moment problem*, Proc. Amer. Math. Soc. **112** (1991)). It gives a similar condition for the solvability of the L-problem of moments, using the operator phase shift. Based on this, it underlines some of the aspects of the operator phase shift used in the L-problem of moments.

0. INTRODUCTION

The L-problem of moments consists of characterizing the moment sequence

$$A_n = \int_{\mathbb{R}} t^n B(t)\, dt, \qquad n \in \mathbb{N}, \tag{0.1}$$

of a measurable operator-valued function $0 \leq B(\lambda) \leq L$.

In the scalar case, this problem was formulated and completely solved by Achiezer and Krein in 1930. The problem can be formulated in the same manner for operator-valued functions. The solvability of the operator L-problem of moments can be linked with the phase shift introduced in the theory of operator perturbation.

The phase shift is a completely unitary invariant that characterizes a perturbation pair of two operators.

This invariant was introduced by Carey in [2]. In this article, it was proved that the principal function $\phi(z) = I + K(A-z)^{-1}K^*$, $\operatorname{Im} z \neq 0$, of the operator pair $\{A, K\}$ admits an exponential representation

$$\Phi(z) = \exp\left(\int_{\mathbb{R}} \frac{B(\lambda)}{\lambda - z}\, d\lambda\right)$$

where $B(\lambda)$ is a summable function taking values in the positive cone of the unit ball of the bounded operators.

This function was called the phase shift. Based on this exponential representation, we gave in [7] a solvability condition for the L-problem of moments.

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In this note, we take again the idea of linking the exponential representation of the phase shift with the L-problem of moments and give another solvability condition.

Further, in the same article, Carey gives a characterization of the spectral type of the operator that was perturbed and also a characterization of the support of the phase shift.

We relate these concepts to the L-problem of moments. Hence, we give a condition in terms of moments which implies that the spectral measures of the appearing operators are absolutely continuous; also we give conditions of the moments which imply that the phase shift has a given support.

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In this note, one of our aims is to prove the equivalent conditions $1° \Leftrightarrow 4° \Leftrightarrow 5°$ from the following theorem (we mention that the equivalent conditions $1° \Leftrightarrow 2° \Leftrightarrow 3°$ were proved in [7]).

Theorem 1.0. *The following assertions are equivalent*:

$1°$. *The sequence* $(A_n)_{n=0}^{\infty}$ *represents the successive moments of a summable operator-valued function,* $0 \le B(\lambda) \le L$.

$2°$. *There is another operator sequence* $(A'_n)_{n=0}^{\infty}$ *for which we have the equality*

$$\exp\left(-L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1}$$

with both operatorial matrices $(A'_{m+n})_{m,n=0}^{\infty}$ *and* $(-A'_{m+k+2} + C_1 A'_{m+k})_{m,k=0}^{\infty}$ *nonnegatively defined for* C_1 *a positive constant.*

$3°$. *There is a spectral measure* $\sigma : \mathrm{Bor}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ *for which*

$$I + \int_{\mathbf{R}} \frac{d\sigma(t)}{t-z} = \exp\left(-L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right).$$

$4°$. *There is another operator sequence* $(A''_n)_{n=0}^{\infty}$ *for which we have the relation*:

$$\exp\left(L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I + \sum_{m=0}^{\infty} A''_m z^{-m-1}$$

with both operatorial matrices $(A''_{n+m})_{n,m=0}^{\infty}$ *and* $(-A''_{m+k+2} + C_2 A''_{m+k})_{m,k=0}^{\infty}$ *nonnegatively defined for* C_2 *a positive constant.*

$5°$. *There is a spectral operator measure* σ': $\mathrm{Bor}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ *for which*:

$$I - \int_{\mathbf{R}} \frac{d\sigma'(t)}{t-z} = \exp\left(L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right).$$

Proof. The equivalent conditions $1° \Leftrightarrow 2° \Leftrightarrow 3°$ were proved in [7]. We shall prove in this note that conditions $1°$, $4°$, and $5°$ are equivalent. We assume first that $1°$ is true; that is, $(A_n)_{n=0}^{\infty}$ represents the sequence of successive moments of a summable operator-valued function $B: \mathbb{R} \to \mathscr{L}(\mathscr{H})$, $0 \le B(\lambda) \le L$, i.e., $A_n = \int_{\mathbf{R}} t^n B(t)\,dt$, $n \in \mathbb{N}$.

From the boundedness of the function B, it implies $0 \le L^{-1}B(t) \le I$, so we have $L^{-1}A_n = \int_{\mathbf{R}} t^n L^{-1}B(t)\,dt$.

The first step is a reduction of the power series of the moments to a Cauchy integral formula. Computing the sum for $\mathbb{N}$-indices we obtain

$$\sum_{n=0}^{\infty} L^{-1}A_n z^{-n-1} = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{t^n}{z^{n+1}} L^{-1}B(t)\,dt = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}} L^{-1}B(t)\,dt$$
$$= \int_{\mathbb{R}} \frac{L^{-1}B(t)}{z-t}\,dt = -\int_{\mathbb{R}} \frac{L^{-1}B(t)}{t-z}\,dt$$

for z sufficiently large.

We shall note that $B'(t) = L^{-1}B(t)$. For this new function, we have $0 \leq B'(t) \leq I$.

Carey's result shows that a summable operator function can be the phase shift of a perturbation pair; i.e., if $\widetilde{B}$ is a $\mathscr{B}(\mathscr{H}, \mathscr{H})$-valued operator function with $0 \leq \widetilde{B} \leq I$, then

$$I + K(A-z)^{-1}K^* = \exp\left(\int_{\mathbb{R}} \frac{\widetilde{B}(t)}{t-z}\,dt\right), \qquad \operatorname{Im} z \neq 0.$$

Instead of z, we take $-z$ in this formula, $\operatorname{Im} -z \neq 0$. With this change, the equality becomes

$$\begin{aligned} I + K(A-(-z))^{-1}K^* &= \exp\left(\int_{\mathbb{R}} \frac{\widetilde{B}(t)}{t-(-z)}\,dt\right) \quad \text{equivalent with} \\ I - K(A'-z)^{-1}K^* &= \exp\left(-\int_{\mathbb{R}} \frac{B'(\lambda)}{\lambda - z}\,d\lambda\right) \end{aligned} \tag{1.1}$$

where we have noted that $A' = -A$, $\lambda = -t$, and $B'(\lambda) = \widetilde{B}(-t)$; because $0 \leq \widetilde{B} \leq I$, that implies $0 \leq B' \leq I$ and $0 \leq B \leq L$. The operator A' is also selfadjoint, and the function B' is also a summable operator function which takes values in the cone of the unit ball of $\mathscr{B}(\mathscr{H}, \mathscr{H})$ operators having

$$\operatorname{supp} B' = -\operatorname{supp} \widetilde{B}. \tag{1.2}$$

For this representation we have

$$\exp\left(-\int_{\mathbb{R}} \frac{B'(\lambda)}{\lambda - z}\,d\lambda\right) = \exp\left(L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - K(A'-z)^{-1}K^* = \tilde{\phi}(z).$$

This exponential representation of the phase shift will help us construct the $(A''_n)_{n=0}^{\infty}$ sequence

$$\tilde{\phi}(z) = I - K(A'-z)^{-1}K^* = I + K\sum_{n=0}^{\infty} \frac{A'^n}{z^{n+1}} K^*.$$

We identify $A''_n = KA'^nK^*$.

From both representations of the principal function $\tilde{\phi}(t)$ we obtain

$$\exp\left(L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I + \sum_{m=0}^{\infty} A''_m z^{-m-1},$$

the required equality.

We shall prove that the obtained matrix $(A''_{n+m})_{n,m=0}^{\infty}$ is nonnegatively defined (i.e., $\sum_{m,k} A''_{m+k} x_m x_k \geq 0$ for every $(x_m)_{m=0}^{\infty}$ sequence with finite support).

With the definition of A''_m, the inequality becomes

$$\sum_{k,m} \langle K A'^{k+m} K^* x_k, x_m \rangle = \sum_{k,m} \langle A'^k K^* x_k, A'^m K^* x_m \rangle = \left\| \sum_{m=0}^{\infty} A'^m K^* x_m \right\|^2 \geq 0.$$

We shall prove in the second turn that there is a constant $C_2 > 0$, so that the matrix $(-A''_{m+k+2} + C_2 A''_{m+k})_{m,k=0}^{\infty}$ is nonnegatively defined.

With the definition of A''_m this condition becomes

$$\sum_{m,k} \langle K A'^{m+k+2} K^* x_m, x_k \rangle \leq \sum_{m,k} C_2 \langle K A'^{m+k} K^* x_m, x_k \rangle.$$

Indeed,

$$\left\| A' \sum_{m=0}^{\infty} A'^m K^* x_m \right\|^2 \leq C_2 \left\| \sum_{m=0}^{\infty} A' K^* x_m \right\|^2,$$

an inequality that is true for $\sqrt{C_2} = \|A'\| > 0$.

We shall prove that 4° *implies* 5°. Suppose that there is an operator sequence $(A''_n)_{n=0}^{\infty}$ for which the matrices $(A''_{n+m})_{n,m=0}^{\infty}$ and $(-A''_{n+k+2} + C_2 A''_{n+k})_{n,k=0}^{\infty}$ are nonnegatively defined and for which we have

$$\exp\left(L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I + \sum_{n=0}^{\infty} A''_n z^{-n-1}.$$

We shall prove the existence of a spectral positive measure with the required property. For this, we consider the operator sequence $(A''_m)_{m=0}^{\infty}$ to be doubly indexed. With the assumption, $(A''_m)_{m=0}^{\infty}$ can be represented as an operator-valued, positively defined function

$$A'' : \mathbb{N} \times \mathbb{N} \to \mathscr{L}(\mathscr{H}), \qquad A''(m, n) = A''_{m+n}.$$

The classical Kolmogorov theorem gives a decomposition for positively defined kernels:

Let $K : I \times I \to \mathscr{L}(\mathscr{H})$ be a positively defined operator-valued function (i.e., $\sum_{i,j} \langle K(i, j) x_i, x_j \rangle \geq 0$ for every family $(x_i)_{i=0}^{\infty}$ with finite support). Then $K(i, j)$ admits a decomposition of the form $K(i, j) = h_i^* h_j$ with $h_i \in \mathscr{L}(\mathscr{H})$.

Thus, A''_{m+n} can be represented as $A''_{n+m} = K_n^* K_m$. From the nonnegativeness condition together with Kolmogorov's decomposition, it follows that we can find a constant $C_2 > 0$ so that $(-K_{n+1}^* K_{m+1} + C_2 K_n^* K_m)_{n,m=0}^{\infty}$ is nonnegatively defined. According to this, for $(x_k)_{k=0}^{\infty}$ an arbitrary family of vectors of finite support, we have

$$C_2 \sum_{m,k} \langle K_m^* K_k x_k, x_m \rangle \geq \sum_{m,k} \langle K_{m+1}^* K_{k+1} x_k, x_m \rangle,$$

an inequality which becomes

$$\sqrt{C_2} \left\| \sum_{k=0}^{\infty} K_k x_k \right\| \geq \left\| \sum_{k=0}^{\infty} K_{k+1} x_k \right\|.$$

We take by definition $A'(\sum_{k=0}^{\infty} K_k x_k) =: \sum_{k=0}^{\infty} K_{k+1} x_k$.

Since the K_n are linear, so is A' and, from our previous remark, A' is continuous. Taking $x_0 = (1, 0, \dots)$, $x_i = 0$ for $i > 1$, we obtain $A'K_0 = K_1$; and using the induction method for a suitable choice of $(x_n)_{n=0}^{\infty}$, we obtain $K_n = A'^n K_0$. We shall prove next that the obtained A' operator is selfadjoint. For x in a dense subset of $\mathscr{H}$, we can find $(x_k)_{k=0}^{\infty}$ such that $x = \sum_{k=0}^{\infty} K_k x_k$. In this case, $\langle A'x, x\rangle = \langle \sum_{k=0}^{\infty} K_{k+1} x_k, \sum_{k=0}^{\infty} K_k x_k\rangle \in \mathbb{R}$ because $K_k^* K_{k+1}$ are positively defined. Thus, $\langle A'x, x\rangle \in \mathbb{R}$ for any $x \in \mathscr{H}$ and so A' is a selfadjoint operator. Because A' is selfadjoint, it admits a representations of the form $A' = \int_{\mathbf{R}} t\, dE(t)$ where $\{E_\lambda\}_\lambda$ is its spectral resolution of the unity. Then we have

$$A''_m = \int_{\mathbf{R}} t^m\, d(K_0^* E(t) K_0) \quad \text{for } m \geq 1 \quad \text{and} \quad A''_0 = K_0^* K_0 = \int_{\mathbf{R}} d\sigma'(t),$$

where we have noted the measure σ': $\mathrm{Bor}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ defined by $\sigma'(\Delta) = K_0^* E(\Delta) K_0$. Considering this measure, the relation from 4° becomes

$$\exp\left(L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - \int_{\mathbf{R}} \frac{d\sigma'(t)}{t - z},$$

the required equality.

We shall prove now that 5° *implies* 1°. Let σ': $\mathrm{Bor}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ be a positive measure, satisfying the equality

$$I - \int_{\mathbf{R}} \frac{d\sigma'(t)}{t - z} = \exp\left(L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1}\right).$$

With the help of this positive measure, we shall construct the function $F(t) = \sigma'(-\infty, t)$, the function that will be shown to be bounded and nondecreasing (i.e., for $t_1 < t_2$, $F(t_1) \leq F(t_2)$); that is, $\sigma'(-\infty, t_2) \leq \sigma'(-\infty, t_1) + \sigma'[t_1, t_2)$. It remains to show that $\sigma'[t_1, t_2) \geq 0$; it means that $\langle \sigma'[t_1, t_2), x, x\rangle \geq 0$, an equality that is true because σ' proceeds from a spectral measure.

We are now able to apply Naimark's dilation theorem [6, Appendix, Theorem 1].

From this theorem, there is a bounded linear mapping K from an auxiliary Hilbert space $\mathscr{H}$ into $\mathscr{X}$ and a resolution of the unity $\{E_\lambda\}_\lambda$ in $\mathscr{H}$ such that $F(\lambda) = K E_\lambda K^*$.

Let A' designate the selfadjoint operator whose resolution of the unity is $\{E_\lambda\}$. We evidently have

$$(1.3) \quad I - \int_{\mathbf{R}} \frac{d\sigma'(t)}{t - z} = I - K(A' - z)^{-1} K^* = \exp\left(L^{-1} \sum_{n=0}^{\infty} A_n z^{-n-1}\right) = \tilde{\phi}(z).$$

From now on, we apply again Carey's result together with the remark in (1.1).

Suppose $\tilde{\phi}(z)$ is the determining function of a perturbation pair $\{A', K\}$ on a Hilbert space $\mathscr{H}$, A' a selfadjoint operator acting on $\mathscr{H}$, and $K: \mathscr{H} \to \mathscr{X}$. Then there is a summable function $B'(\lambda)$ with values in the set of the positive operators of the unit ball of $\mathscr{B}(\mathscr{H}, \mathscr{H})$ such that

$$\tilde{\phi}(z) = \exp\left(-\int_{\mathbf{R}} \frac{B'(t)}{t - z}\, dt\right) = I - K(A' - z)^{-1} K^*.$$

From both representations (1.1) and (1.3) we have

$$L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1} = \sum_{n=0}^{\infty}\left(\int_{\mathbf{R}} t^n B'(t)\,dt z^{-n-1}\right),$$

that is, $A_n = \int_{\mathbf{R}} Lt^n B'(t)\,dt$, $n \in \mathbb{N}$.

By noting $\widetilde{B}(t) = LB'(t)$, we obtain

$$A_n = \int_{\mathbf{R}} t^n B(t)\,dt, \qquad n \in \mathbb{N}.$$

This assertion completes the proof of the theorem.

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Our next result is based on the previous theorem. We give a characterization of the support of the function $B(\lambda)$ and a necessary and sufficient condition for A and A' operators to be absolutely continuous. These properties are expressed in terms of moments.

Proposition. *The solution of the moment problem $B(\lambda)$ has the support included in $[m, M]$ if and only if the matrices $(A'_{n+k+1} - mA'_{n+k})_{n,k=0}^{\infty}$ and $(MA''_{n+k} - A''_{n+k+1})_{n,k=0}^{\infty}$ are nonnegatively defined. ($(A'_n)_{n=0}^{\infty}$ and $(A''_m)_{m=0}^{\infty}$ are the sequences obtained from the previous theorem.)*

Remark. The consideration about the support is true in the case that we have only the lower or the upper bound of it; in that case it remains only one nonnegativeness condition for the two matrices.

Proof. If we have $\operatorname{supp} B(\lambda) \subseteq [m, +\infty)$, from Carey's Proposition 4.1 [2], $\operatorname{supp}\sigma = \sigma(A) \subseteq \operatorname{supp} B(\lambda) \subseteq [m, +\infty)$. We shall prove the nonnegativeness of the matrix $(A'_{n+k+1} - mA'_{n+k})_{n,k=0}^{\infty}$; that is,

$$\begin{aligned}
&\sum_{n,k=0}^{\infty} \langle (A'_{n+k+1} - mA'_{n+k})x_n, x_k\rangle \\
&\quad = \sum_{n,k=0}^{\infty} \left\langle \int_{\mathbf{R}} (t^{n+k+1} - mt^{n+k})d\sigma(t)x_n, x_k \right\rangle \\
&\quad = \int_{\mathbf{R}} (t-m)d\left\langle \sigma(t)\sum_{n=0}^{\infty} t^n x_n, \sum_{k=0}^{\infty} t^k x_k \right\rangle \geq 0 \quad (\text{for } t = \bar{t} \text{ on } \mathbb{R}).
\end{aligned}$$

Conversely. We have $\sum_{n,k=0}^{\infty}\langle (A'_{n+k+1} - mA'_{n+k})x_n, x_k\rangle \geq 0$ for every $(x_n)_{n=0}^{\infty} \in \mathscr{H}$ with finite support; that is,

$$\int_{\mathbf{R}} (t-m)d\left\langle \sigma(t)\sum_{n=0}^{\infty} t^n x_n, \sum_{n=0}^{\infty} t^n x_n \right\rangle \geq 0.$$

This inequality implies that $t \geq m$, equivalent with $\operatorname{supp}\sigma \subseteq [m, +\infty)$.

From Carey's Proposition 4.1 [2], $\operatorname{supp}\sigma \subseteq \operatorname{supp} B(\lambda)$, an inclusion equivalent with $\operatorname{supp} B(\lambda) \subseteq [m, +\infty)$.

With the construction in Theorem 1.0, remark (1.2), $B'(t) = \widetilde{B}(-t)$; if $\operatorname{supp}\widetilde{B}(t) \subseteq [-M, +\infty)$, then $\operatorname{supp} B'(t) \subseteq (-\infty, M]$ and by Carey's result

$\operatorname{supp}\sigma' \subseteq \operatorname{supp} B'(\lambda) = \operatorname{supp} B(\lambda) \subseteq [-\infty, M]$. We shall prove the nonnegativeness of the matrix $(MA''_{n+k} - A''_{n+k+1})_{n,k=0}^{\infty}$; that is,

$$\begin{aligned}
&\sum_{n,k=0}^{\infty} \langle (MA''_{n+k} - A''_{n+k+1})x_n, x_k\rangle \\
&\quad = \sum_{n,k=0}^{\infty} \left\langle \int_{\mathbf{R}} (Mt^{n+k} - t^{n+k+1})\, d\sigma'(t) x_n, x_k \right\rangle \\
&\quad = \int_{\mathbf{R}} (M-t) d\sigma' \left\langle (t) \sum_{n=0}^{\infty} t^n x_n, \sum_{k=0}^{\infty} t^k x_k \right\rangle \geq 0.
\end{aligned}$$

Conversely. If $(MA''_{n+k} - A''_{n+k+1})_{n,k=0}^{\infty}$ is positively defined, it implies that for every $(x_n) \subset \mathscr{H}$ with finite support,

$$\int_{\mathbf{R}} (M-t) d\sigma' \left\langle (t) \sum_{n=0}^{\infty} t^n x_n, \sum_{n=0}^{\infty} t^n x_n \right\rangle \geq 0,$$

that is, $t \leq M$, equivalent with $\operatorname{supp}\sigma' \subseteq (-\infty, M]$ and so $\operatorname{supp} B'(t) \subseteq (-\infty, M]$; $\operatorname{supp} B(t) = \operatorname{supp} B'(t) \subseteq (-\infty, M]$.

We consider now the case in which B has a compact support; we define in this case the function $\widetilde{I}\colon \mathbb{R} \to \mathscr{B}(\mathscr{X}, \mathscr{X})$ by

$$\widetilde{I}(t) = \begin{cases} LI(t) & \text{for } t \in \operatorname{supp} B, \\ 0 & \text{for } t \in C \operatorname{supp} B. \end{cases}$$

This function is summable, with values in the set of positive operators of the L-ball of $\mathscr{B}(\mathscr{X}, \mathscr{X})$. In this case, $L^{-1}\widetilde{I}$ can be the phase shift of a perturbation pair of two operators. We shall define the L-moments of this function, i.e.,

$$b_n = \int_{\mathbf{R}} t^n \widetilde{I}(t)\, dt, \qquad n \in \mathbb{N}$$

(the integral exists because $\widetilde{I}$ has a compact support).

In [2] the upper right numerical oscillation of $B(\lambda)$ on (a, b) was introduced by setting:

$$W(B, a, b) = \sup\{([B(\lambda'') - B(\lambda')]y, y),\ a < \lambda' < \lambda'' < b,\ \|y\| = 1\}$$

and

$$W(B, \lambda) = \lim_{a\uparrow\lambda,\, b\downarrow\lambda} W(B, a, b).$$

With these definitions, Carey proved in [2, Theorem 4.6] the following:

Let $\{V, K\}$ be a perturbation pair with phase $B(\lambda)$. The operator V is absolutely continuous on the open set $E_B = \{\lambda | W(\lambda, B) < 1\}$.

In this paper, with the help of the numerical oscillation of $B(\lambda)$, the phase shift used in Theorem 1.0, we introduce the functions

$$s_{\lambda_0}(t) = \begin{cases} W(\lambda_0, B)I(t) & \text{for } t \in \operatorname{supp} B, \\ 0 & \text{for } t \in C \operatorname{supp} B \end{cases}$$

for every $\lambda_0 \in \mathbb{R}$.

These functions are summable, taking values in the set of positive operators of the L-ball of $\mathscr{B}(\mathscr{X}, \mathscr{X})$ and having compact support.

For them, there exist the moments:

$$a^{(\lambda_0)} = \int_{\mathbf{R}} t^n s_{\lambda_0}(t)\, dt \qquad \text{for } n \in \mathbb{N}.$$

By using the introduced notions, we have the following proposition:

Proposition. *The operators A and A' from Theorem* 1.0 *are absolutely continuous on the set M if and only if $\sum_{n,k=0}^{\infty} \langle (b_{n+k} - a_{n+k}^{(\lambda_0)}) w_n, w_k \rangle > 0$ for every $\lambda_0 \in M \cap \operatorname{supp} B$ and $\sum_{n,k=0}^{\infty} (b_{n+k} - a_{n+k}^{(\lambda_0)}) w_n \overline{w}_k = 0$ for $\lambda_0 \in C \operatorname{supp} B \cap M$.*

The proof of this Proposition is evident; it is based only on Carey's result [2, Theorem 4.6].

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