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SOME COUNTEREXAMPLES TO THE REGULARITY OF MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We present examples to show that the solution u of the Monge-Ampère equation $\det(D^2u) = f(x)$, with $u = 0$ on the boundary, may not lie in $W^{2,p}$ or in $C^{1,\alpha}$ for noncontinuous and positive $f(x)$ and for continuous and nonnegative $f(x)$.

Recently L. A. Caffarelli proved interior $W^{2,p}$ estimates and derivative Hölder estimates for viscosity solutions of the Monge-Ampère equation

$$\begin{cases} M(u) = \det(D^2u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a convex domain in $\mathbb{R}^n$ with $B_1 \subset \Omega \subset B_n$ and f is bounded and nonnegative. His main results are as follows:

Proposition. (1) *If f is strictly positive, then u is strictly convex and $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.*

(2) *If, moreover, f is continuous, then $u \in W^{2,p}(B_{1/2})$ for any $p > 1$.*

(3) *If $f \in C^\alpha$, then $u \in C^{2,\alpha}$.*

The purpose of this paper is to present some examples to show that in the Proposition above, if f is not continuous, u may fail to be of class $W^{2,p}$; if f is not strictly positive, u may fail to be $C^{1,\alpha}$ smooth for any $\alpha > 0$, even though $f(x)$ is continuous.

We discuss weak solutions only. It is indicated by Caffarelli that a weak solution is also a viscosity solution for Monge-Ampère equations and that, for continuous $f(x)$, they are equivalent. Our examples depend on the following simple property of Monge-Ampère equations.

Lemma 1. *Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^n$ with disjoint interiors. Suppose u_1 and u_2 are convex and verify $M(u) = f(x)$ in Ω_1 and Ω_2, respectively. If $u_1 = u_2$, $Du_1 = Du_2$ on $\partial\Omega_1 \cap \partial\Omega_2$, then*

$$u = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2, \end{cases}$$

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is convex and u *is a weak solution of* $M(u) = f(x)$ *in the interior* Ω *of* $\overline{\Omega_1 \cup \Omega_2}$.

Example 1. Let $u = x^4 + \frac{3}{2}y^2/x^2$ if $|y| \le |x|^3$ and $u = \frac{1}{2}x^2y^{2/3} + 2y^{4/3}$ if $|y| > |x|^3$, and let $\Omega = \{(x, y) \in \mathbb{R}^2,\ u(x, y) < 1\}$. Then $\frac{1}{3} \le M(u) \le 36$ in Ω, but u is not in $W^{2,p}$ for $p > 2$.

Proof. It is easy to see that $u \in C^1(\overline{\Omega})$ and u is strictly convex in Ω. Direct computation gives

$$36 \ge M(u) = 36 - 9y^2x^{-6} \ge 27 \quad \text{if } |y| \le |x|^3,$$
$$\frac{8}{9} \ge M(u) = \frac{8}{9} - \frac{5}{9}x^2y^{-2/3} \ge \frac{1}{3} \quad \text{if } |y| \ge |x|^3.$$

Since $|\operatorname{Image}(Du)|_{B_r} \to 0$ as $r \to 0$, we therefore conclude that $\frac{1}{3} \le M(u) \le 36$. But obviously $u \notin W^{2,p}$ for $p > 2$.

If we let

$$u = x^\alpha + \frac{\alpha^2 - 1}{\alpha(\alpha - 2)} y^2 x^{2-\alpha}$$

for $|y| \le |x|^{\alpha-1}$ and

$$u = \frac{1}{2\alpha} x^2 y^{(\alpha-2)/(\alpha-1)} + \frac{4\alpha - 5}{2(\alpha - 2)} y^{\alpha/(\alpha-1)}$$

for $|y| > |x|^{\alpha-1}$, then u is strictly convex and $0 < C_1(\alpha) \le M(u) \le C_2(\alpha)$, but $u \notin W^{2,p}$ for $p > \frac{\alpha}{\alpha-2}$.

The above example shows that for positive and noncontinuous $f(x)$, the solution u of problem (1) may fail to lie in $W^{2,p}$. In the following we discuss the degenerate case $f(x) \ge 0$.

Example 2. Let

$$u = \varphi(x, y) = 4x^2e^{-1/|x|} + x^2e^{+1/|x|}y^2 \quad \text{if } |y| < h(x)e^{-1/|x|},$$

and let

$$u = \psi(x, y) = 3g(x)x^2e^{-1/|x|} + 2|y|\log^{-2}|y| \quad \text{if } |y| \ge h(x)e^{-1/|x|}.$$

Then by proper choice of g and h, we claim that $u \in C^1(B_r)$, u is strictly convex in B_r, and $0 \le M(u) \le C$ in B for some r small. But $u \notin C^{1,\alpha}(B_r)$ for any $\alpha > 0$.

Proof. Let g and h be even functions such that $\varphi = \psi$ and $\varphi_y = \psi_y$ on $\{|y| = h(x)e^{-1/|x|}\}$ (which automatically makes $\varphi_x = \psi_x$ on $\{|y| = h(x)e^{-1/|x|}\}$), that is,

$$4 + h^2 = 3g + 2h(1 - |x|\log h)^{-2}, \qquad h = (1 - |x|\log h)^{-2} + 2|x|(1 - |x|\log h)^{-3}.$$

To get the even function h we let $h(0) = 1$, then differentiate

$$h = (1 - x\log h)^{-2} + 2x(1 - x\log h)^{-3},$$

and obtain, by the existence theory of differential equations, a local analytic solution $h(x)$. From the first formula above we then obtain $g(x)$ which is also analytic on some interval $[0, \delta]$ with $g(0) = 1$. For r small enough, we have

$$0 \le M(\varphi) \le C \quad \text{in } B_r \cap \{|y| < h(x)e^{-1/|x|}\}$$

and

$$0 \le M(\psi) \le Cx \quad \text{in } B_r \cap \{|y| > h(x)e^{-1/|x|}\},$$

where C depends on g, h. Hence $u \in C^1(B_r)$ and u is strictly convex with $M(u) \le C$ in B_r. But $u \notin C^{1,\alpha}(B_r)$ for any $\alpha > 0$.

But for the function u given in Example 2, we find $f(x) = M(u)$ is not continuous. In the case when $f(x)$ is continuous, we have the following example.

Example 3. Let Ω be the unit ball in $\mathbb{R}^n$, and let

$$f(x) = \eta(x_n/|x'|^\alpha)|x'|^\beta, \qquad x' = (x_1, \dots, x_{n-1}),$$

where $\alpha > 1$, $\beta > 0$, and

$$\eta(t) = \begin{cases} e^{-1/(1-t^2)}, & |t| < 1, \\ 0, & |t| \ge 1. \end{cases}$$

Then the solution u of problem (1) is not C^1 smooth if $\beta < (n-2)\alpha - 2(n-1)$ and u is not $C^{1,1}$ smooth if $\beta < 2(n-1)(\alpha - 1)$.

Proof. First note that the solution u is axially symmetric in the x_n direction and that $u(x', x_n) = u(x', -x_n)$, since if $u(x', x_n)$ is a solution of (1), then for any orthogonal transformation T from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$, $u(Tx', x_n)$ is also a solution of (1), but the solution is unique. It therefore follows that u reaches its strict minimum at the origin.

Set $\varphi(t) = |t|^\alpha$, $\psi(t) = |t|^\beta$, and let $h(t) = u(0', t)$. Obviously $h(t)$ is convex for $t \in (-1, 1)$. If $h(t)$ is strictly convex at $t_0 \in (0, 1)$, we claim

$$u(0', t_0) = u(x', t_0) \quad \text{for any } |x'| = \varphi^{-1}(t_0) = t_0^{1/\alpha}.$$

Indeed, if $u(0, t_0) < u(x', t_0)$ for $|x'| = \varphi^{-1}(t_0)$, let

$$u^*(x', x_n) = u(x', x_n) - D_{x_n}u(0', t_0)(x_n - t_0).$$

Then $\{u^*(x) < \min u^* + \varepsilon\} \subset \{x_n > \varphi(|x'|)\}$ for $\varepsilon > 0$ small enough. But $M(u^*) = f = 0$ in $\{x_n > \varphi(|x'|)\}$, a contradiction.

To prove the conclusion of Example 3 we may suppose $u(x) \in C^1(\Omega)$, otherwise we are through. In this case we have $h \in C^1(-1, 1)$. For any $\varepsilon_0 > 0$ small, since $h(t)$ reaches its minimum at $t = 0$, there must exist $\varepsilon \in (0, \varepsilon_0)$ so that $h(t)$ is strictly convex at ε. We will show that $u(0', \varepsilon) = h(\varepsilon) \ge C\varepsilon^{(2\alpha+2(n-1)+\beta)/n\alpha}$ for some $C > 0$ depending only on α, β, and n. Hence, $u(x)$ is not C^1 smooth provided $(2\alpha + 2(n-1) + \beta)/n\alpha < 1$, i.e., $\beta < (n-2)\alpha - 2(n-1)$, and if $(2\alpha + 2(n-1) + \beta)/n\alpha < 2$, i.e., $\beta < 2(n-1)(\alpha - 1)$, then u is not $C^{1,1}$ smooth.

For the point $t = \varepsilon > 0$ where $h(t)$ is strictly convex, let d_ε be the number so that $u(x', 0)|_{|x'|=d_\varepsilon} = u(0', \varepsilon)$. By convexity we have $d_\varepsilon \ge \varphi^{-1}(\varepsilon) = \varepsilon^{1/\alpha}$. Let T denote the transformation $(y', y_n) = T(x', x_n) = (x'/d_\varepsilon, x_n/\varepsilon)$, and let

$$w(y', y_n) = [u(d_\varepsilon y', \varepsilon y_n) - u(0, 0)]/[\varepsilon d_\varepsilon^{n-1}]^{2/n}.$$

Then $w(y) \ge 0$ and

$$\det(D^2 w) = f(d_\varepsilon y', \varepsilon y_n) \quad \text{in } \Omega_\varepsilon = T(\{u(x) < u(0', \varepsilon)\}).$$

Obviously $B_{1/\sqrt{n}} \subset \Omega_\varepsilon$, and $w = \text{const}$ on $\partial\Omega_\varepsilon$.

Set $M_\varepsilon = \sup\{w(y), y \in \Omega_\varepsilon\}$. By convexity we have $w(x) < M_\varepsilon/2$ and $|Dw(x)| \le |\sup_{\Omega_\varepsilon} w - w(x)| / \operatorname{dist}(x, \partial\Omega_\varepsilon) \le 2\sqrt{n} M_\varepsilon$ on $\partial(\frac{1}{2}\Omega_\varepsilon)$, where $\frac{1}{2}\Omega_\varepsilon = \{\frac{1}{2}y ; y \in \Omega_\varepsilon\}$. Hence,

$$\int_{\frac{1}{2}\Omega_\varepsilon} f(d_\varepsilon y', \varepsilon y_n)\, dy = \int_{\frac{1}{2}\Omega_\varepsilon} \det(D^2 w)\, dy = \operatorname{mes}\{Dw(y) ; y \in \frac{1}{2}\Omega_\varepsilon\} \le C_1 M_\varepsilon^n .$$

We thus obtain

$$\begin{aligned} M_\varepsilon &\ge \left[\frac{1}{C_1}\int_{\frac{1}{2}\Omega_\varepsilon} f(d_\varepsilon y', \varepsilon y_n)\, dy\right]^{1/n} = \left[\frac{1}{C_1 \varepsilon d_\varepsilon^{n-1}} \int_{T^{-1}(\frac{1}{2}\Omega_\varepsilon)} f(x)\, dx\right]^{1/n} \\ (*) \qquad &\ge \left[\frac{1}{C_1 \varepsilon d_\varepsilon^{n-1}} \operatorname{mes}(D) \cdot \min_D f(x)\right]^{1/n} \ge C_2 \left[\min_D f(x)\right]^{1/n} \\ &\ge C_3[\psi(\tfrac{1}{2}\varphi^{-1}(\tfrac{1}{8}\varepsilon))]^{1/n} = C_3' \varepsilon^{\beta/\alpha n} , \end{aligned}$$

where

$$D = T^{-1}(\tfrac{1}{2}\Omega_\varepsilon) \cap \{|x'| > \tfrac{1}{2}\varphi^{-1}(\tfrac{1}{8}\varepsilon),\ x_n < \tfrac{1}{2}\varphi(|x'|)\} .$$

Verification of the inequality $(*)$. If $d_\varepsilon \le 4\varphi^{-1}(\varepsilon) = 4\varepsilon^{1/\alpha}$, then by

$$\begin{aligned} D &\supset \{\tfrac{1}{2}\varphi^{-1}(\tfrac{1}{8}\varepsilon) < |x'| < \tfrac{1}{2}\varphi^{-1}(\varepsilon),\ |x_n| < \tfrac{1}{16}\varepsilon\} \\ &= \{\tfrac{1}{2}(\tfrac{1}{8}\varepsilon)^{1/\alpha} < |x'| < \tfrac{1}{2}\varepsilon^{1/\alpha},\ |x_n| < \tfrac{1}{16}\varepsilon\} \end{aligned}$$

we have $\operatorname{mes}(D) \ge C\varepsilon d_\varepsilon^{n-1}$. If $d_\varepsilon > 4\varphi^{-1}(\varepsilon)$, then by the convexity of Ω_ε we have

$$\begin{aligned} D &\supset \{\tfrac{1}{2}\varphi^{-1}(\varepsilon) < |x'| < \tfrac{1}{4}[\varphi^{-1}(\varepsilon) + d_\varepsilon],\ |x_n| < \tfrac{1}{4}\varepsilon\} \\ &\supset \{\tfrac{1}{8}d_\varepsilon < |x'| < \tfrac{1}{4}d_\varepsilon,\ |x_n| < \tfrac{1}{3}\varepsilon\} , \end{aligned}$$

which also implies $\operatorname{mes}(D) \ge C\varepsilon d_\varepsilon^{n-1}$.

We therefore conclude

$$\begin{aligned} (**) \qquad u(0, \varepsilon) - u(0) &= (\varepsilon d_\varepsilon^{n-1})^{2/n} M_\varepsilon \ge (\varepsilon \cdot \varepsilon^{(n-1)/\alpha})^{2/n} \cdot C_3' \varepsilon^{\beta/\alpha n} \\ &\ge C_3' \varepsilon^{(2\alpha + 2(n-1) + \beta)/n\alpha} . \end{aligned}$$

If

$$(2\alpha + 2(n-1) + \beta)/n\alpha < 1 ,$$

i.e., $\beta < (n-2)\alpha - 2(n-1)$, then from the inequality $(**)$ we see that $u(x)$ cannot be C^1 continuous. If

$$(2\alpha + 2(n-1) + \beta)/n\alpha < 2 ,$$

i.e., $\beta < 2(n-1)(\alpha - 1)$, then the inequality $(**)$ shows that $u(x)$ cannot be $C^{1,1}$ smooth. This completes the proof.

Remark 1. In Example 3, since $\alpha > 1$, we have $f(x) \in C^{\beta/\alpha}(\overline{\Omega})$. Let $\beta = 2(n-1)(\alpha-1) - 1$, and let $\alpha > 1$ be large enough. Then $f \in C^{2(n-1)-\delta}$ for some $\delta > 0$ small, but the solution u is not $C^{1,1}$ smooth. On the other hand, by the concavity of $\det^{1/n}(D^2 u)$ and by the Alexandrov maximum principle one sees that if $f^{1/n} \in C^{1,1}(\overline{\Omega})$ and if $\partial\Omega$ is C^2 smooth and strictly convex, then the solution u of the problem (1) is $C^{1,1}$ smooth.

Remark 2. In [W] we proved the following C^2 regularity for the solution u of the problem (1).

Theorem 1. *Let* $w(r) = \sup\{|f(x) - f(y)|\,;\ |x-y| \le r\}$. *If* f *is positive and* $\int_0^1 \frac{\omega(r)}{r}\,dr < \infty$, *then* $u \in C^2(B_{1/2})$.

Note that the function $u(x, y) = x^2/\log|\log r^2| + y^2 \log|\log r^2|$, where $r^2 = x^2 + y^2$, is strictly convex and $M(u) = 4 + O(\log|\log r^2|/\log r^2)$ near the origin, but that $u \notin W^{2,\infty}$. Hence, for continuous and positive $f(x)$, the solution u of equation (1) may fail to lie in $W^{2,\infty}$.

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