SOME COUNTEREXAMPLES TO THE REGULARITY OF MONGE-AMPÈRE EQUATIONS

XU-JIA WANG

(Communicated by Barbara Lee Keyfitz)

Abstract. We present examples to show that the solution $u$ of the Monge-Ampère equation $\det(D^2u) = f(x)$, with $u = 0$ on the boundary, may not lie in $W^{2,p}$ or in $C^{1,\alpha}$ for noncontinuous and positive $f(x)$ and for continuous and nonnegative $f(x)$.

Recently L. A. Caffarelli proved interior $W^{2,p}$ estimates and derivative Hölder estimates for viscosity solutions of the Monge-Ampère equation

$$\begin{cases} M(u) = \det(D^2u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega$ is a convex domain in $\mathbb{R}^n$ with $B_1 \subset \Omega \subset B_n$ and $f$ is bounded and nonnegative. His main results are as follows:

Proposition. (1) If $f$ is strictly positive, then $u$ is strictly convex and $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.

(2) If, moreover, $f$ is continuous, then $u \in W^{2,p}(B_{1/2})$ for any $p > 1$.

(3) If $f \in C^\alpha$, then $u \in C^{2,\alpha}$.

The purpose of this paper is to present some examples to show that in the Proposition above, if $f$ is not continuous, $u$ may fail to be of class $W^{2,p}$; if $f$ is not strictly positive, $u$ may fail to be $C^{1,\alpha}$ smooth for any $\alpha > 0$, even though $f(x)$ is continuous.

We discuss weak solutions only. It is indicated by Caffarelli that a weak solution is also a viscosity solution for Monge-Ampère equations and that, for continuous $f(x)$, they are equivalent. Our examples depend on the following simple property of Monge-Ampère equations.

Lemma 1. Let $\Omega_1$ and $\Omega_2$ be two domains in $\mathbb{R}^n$ with disjoint interiors. Suppose $u_1$ and $u_2$ are convex and verify $M(u) = f(x)$ in $\Omega_1$ and $\Omega_2$, respectively. If $u_1 = u_2$, $Du_1 = Du_2$ on $\partial \Omega_1 \cap \partial \Omega_2$, then

$$u = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2, \end{cases}$$
is convex and \( u \) is a weak solution of \( M(u) = f(x) \) in the interior \( \Omega \) of \( \Omega_1 \cup \Omega_2 \).

**Example 1.** Let \( u = x^4 + \frac{3}{2} y^2/2^2 \) if \( |y| \leq |x|^3 \) and \( u = \frac{1}{2} x^2 y^{2/3} + 2 y^{4/3} \) if \( |y| > |x|^3 \), and let \( \Omega = \{(x, y) \in \mathbb{R}^2, u(x, y) < 1\} \). Then \( \frac{1}{3} \leq M(u) \leq 36 \) in \( \Omega \), but \( u \) is not in \( W^{2,p} \) for \( p > 2 \).

**Proof.** It is easy to see that \( u \in C^1(\overline{\Omega}) \) and \( u \) is strictly convex in \( \Omega \). Direct computation gives

\[
36 \geq M(u) = 36 - 9 y^2 x^{-6} \geq 27 \quad \text{if} \ |y| < |x|^3, \\
\frac{8}{3} \geq M(u) = \frac{8}{3} x^2 y^{-2/3} \geq \frac{1}{3} \quad \text{if} \ |y| > |x|^3.
\]

Since \( |\text{Image}(Du)|_{B_r} \to 0 \) as \( r \to 0 \), we therefore conclude that \( \frac{1}{3} \leq M(u) \leq 36 \). But obviously \( u \notin W^{2,p} \) for \( p > 2 \).

If we let

\[
u = x^\alpha + \frac{\alpha^2 - 1}{\alpha(\alpha - 2)} y^2 x^{2-\alpha}
\]

for \( |y| \leq |x|^{\alpha-1} \) and

\[
u = \frac{1}{2\alpha} x^2 y^{(\alpha-2)/(\alpha-1)} + \frac{4\alpha - 5}{2(\alpha - 2)} y^{\alpha/(\alpha-1)}
\]

for \( |y| > |x|^{\alpha-1} \), then \( u \) is strictly convex and \( 0 < C_1(\alpha) \leq M(u) \leq C_2(\alpha) \), but \( u \notin W^{2,p} \) for \( p > \frac{\alpha}{\alpha-2} \).

The above example shows that for positive and noncontinuous \( f(x) \), the solution \( u \) of problem (1) may fail to lie in \( W^{2,p} \). In the following we discuss the degenerate case \( f(x) > 0 \).

**Example 2.** Let

\[
u = \phi(x, y) = 4 x^2 e^{-1/|x|} + x^2 e^{1/|x|} y^2 \quad \text{if} \ |y| < h(x)e^{-1/|x|},
\]

and let

\[
u = \psi(x, y) = 3 g(x) x^2 e^{-1/|x|} + 2 |y| \log^{-2} |y| \quad \text{if} \ |y| \geq h(x)e^{-1/|x|}.
\]

Then by proper choice of \( g \) and \( h \), we claim that \( u \in C^1(B_r) \), \( u \) is strictly convex in \( B_r \), and \( 0 \leq M(u) \leq C \) in \( B \) for some \( r \) small. But \( u \notin C^{1,\alpha}(B_r) \) for any \( \alpha > 0 \).

**Proof.** Let \( g \) and \( h \) be even functions such that \( \phi = \psi \) and \( \phi_y = \psi_y \) on \( \{|y| = h(x)e^{-1/|x|}\} \) (which automatically makes \( \phi_x = \psi_x \) on \( \{|y| = h(x)e^{-1/|x|}\} \)), that is,

\[
4 + h^2 = 3 g + 2 h (1 - |x| \log h)^{-2}, \quad h = (1 - |x| \log h)^{-2} + 2 |x| (1 - |x| \log h)^{-3}.
\]

To get the even function \( h \) we let \( h(0) = 1 \), then differentiate

\[
h = (1 - x \log h)^{-2} + 2 x (1 - x \log h)^{-3},
\]

and obtain, by the existence theory of differential equations, a local analytic solution \( h(x) \). From the first formula above we then obtain \( g(x) \) which is also analytic on some interval \([0, \delta]\) with \( g(0) = 1 \). For \( r \) small enough, we have

\[
0 \leq M(\psi) \leq C \quad \text{in} \ B_r \cap \{|y| < h(x)e^{-1/|x|}\}.
\]
and

\[ 0 \leq M(\psi) \leq C \quad \text{in } B_r \cap \{|y| > h(x)e^{-1/|x|}\}, \]

where \( C \) depends on \( g, h \). Hence \( u \in C^1(B_r) \) and \( u \) is strictly convex with \( M(u) \leq C \) in \( B_r \). But \( u \notin C^{1, \alpha}(B_r) \) for any \( \alpha > 0 \).

But for the function \( u \) given in Example 2, we find \( f(x) = M(u) \) is not continuous. In the case when \( f(x) \) is continuous, we have the following example.

**Example 3.** Let \( \Omega \) be the unit ball in \( \mathbb{R}^n \), and let

\[ f(x) = \eta(x_n/|x'|^\alpha)|x'|^\beta, \quad x' = (x_1, \ldots, x_{n-1}), \]

where \( \alpha > 1, \beta > 0 \), and

\[ \eta(t) = \begin{cases} e^{-1/(t-\frac{1}{2})}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases} \]

Then the solution \( u \) of problem (1) is not \( C^1 \) smooth if \( \beta < (n-2)\alpha - 2(n-1) \) and \( u \) is not \( C^{1,1} \) smooth if \( \beta < 2(n-1)(\alpha-1) \).

**Proof.** First note that the solution \( u \) is axially symmetric in the \( x_n \) direction and that \( u(x', x_n) = u(x', -x_n) \), since if \( u(x', x_n) \) is a solution of (1), then for any orthogonal transformation \( T \) from \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^{n-1} \), \( u(Tx', x_n) \) is also a solution of (1), but the solution is unique. It therefore follows that \( u \) reaches its strict minimum at the origin.

Set \( \varphi(t) = |t|^{\alpha}, \psi(t) = |t|^\beta \), and let \( h(t) = u(0', t) \). Obviously \( h(t) \) is convex for \( t \in (-1, 1) \). If \( h(t) \) is strictly convex at \( t_0 \), we claim

\[ u(0', t_0) = u(x', t_0) \quad \text{for any } |x'| = \varphi^{-1}(t_0) = t_0^{1/\alpha}. \]

Indeed, if \( u(0, t_0) < u(x', t_0) \) for \( |x'| = \varphi^{-1}(t_0) \), let

\[ u^*(x', x_n) = u(x', x_n) - D_{x_n}u(0, t_0)(x_n - t_0). \]

Then \( \{u^*(x) < \min u^* + \varepsilon \} \subset \{x_n > \varphi(|x'|)\} \) for \( \varepsilon > 0 \) small enough. But \( M(u^*) = f = 0 \) in \( \{x_n > \varphi(|x'|)\} \), a contradiction.

To prove the conclusion of Example 3 we may suppose \( u(x) \in C^1(\Omega) \), otherwise we are through. In this case we have \( h \in C^1(-1, 1) \). For any \( \varepsilon_0 > 0 \) small, since \( h(t) \) reaches its minimum at \( t = 0 \), there must exist \( \varepsilon \in (0, \varepsilon_0) \) so that \( h(t) \) is strictly convex at \( \varepsilon \). We will show that \( u(0', \varepsilon) = h(\varepsilon) \geq Ce^{(2\alpha+2(n-1)+\beta)/n\alpha} \) for some \( C > 0 \) depending only on \( \alpha, \beta, \) and \( n \). Hence, \( u(x) \) is not \( C^1 \) smooth provided \( (2\alpha + 2(n-1) + \beta)/n\alpha < 1 \), i.e., \( \beta < (n-2)\alpha - 2(n-1) \), and if \( (2\alpha + 2(n-1) + \beta)/n\alpha < 2 \), i.e., \( \beta < 2(n-1)(\alpha-1) \), then \( u \) is not \( C^{1,1} \) smooth.

For the point \( t = \varepsilon > 0 \) where \( h(t) \) is strictly convex, let \( \varepsilon_\theta \) be the number so that \( u(x', 0)|_{|x'|=\varepsilon_\theta} = u(0', \varepsilon) \). By convexity we have \( \varepsilon_\theta \geq \varphi^{-1}(\varepsilon) = \varepsilon^{1/\alpha} \).

Let \( T \) denote the transformation \( (y', y_n) = T(x', x_n) = (x'/\varepsilon_\theta, x_n/\varepsilon) \), and let

\[ w(y', y_n) = [u(\varepsilon_\theta y', \varepsilon y_n) - u(0, 0)]/[\varepsilon d_{\theta}^{n-1}]^{2/n}. \]

Then \( w(y) \geq 0 \) and

\[ \det(D^2w) = f(\varepsilon_\theta y', \varepsilon y_n) \quad \text{in } \Omega_\varepsilon = T(\{u(x) < u(0', \varepsilon)\}). \]

Obviously \( B_{1/\sqrt{n}} \subset \Omega_\varepsilon \), and \( w = \text{const} \) on \( \partial \Omega_\varepsilon \).
Set \( M_e = \sup \{ w(y), y \in \Omega_e \} \). By convexity we have \( w(x) < M_e/2 \) and \( |Du(x)| \leq \frac{1}{2} \) for \( x \in \partial \Omega_e \), where \( \Omega_e = \{ \frac{1}{2} y; y \in \Omega_e \} \). Hence,

\[
\int_{\frac{1}{2} \Omega_e} f(dx') \, dy = \int_{\frac{1}{2} \Omega_e} \det(D^2 w) \, dy = \text{mes}(Dw(y); y \in \Omega_e) \leq C_1 M_e^n.
\]

We thus obtain

\[
M_e \geq \left[ \frac{1}{C_1} \int_{\frac{1}{2} \Omega_e} f(dx') \, dy \right]^{1/n} = \left[ \frac{1}{C_1 \varepsilon e^{d e^{-1}}} \int_{T^{-1}(\frac{1}{2} \Omega_e)} f(x) \, dx \right]^{1/n}
\]

(\#)

\[
\geq \left[ \frac{1}{C_1 \varepsilon e^{d e^{-1}}} \text{mes}(D) \cdot \min_D f(x) \right]^{1/n} \geq C_2 \left[ \min_D f(x) \right]^{1/n}
\]

\[
\geq C_3 \left[ \frac{1}{2} \varepsilon^{-1} (\frac{1}{2} \varepsilon) \right]^{1/n} = C_3 \varepsilon^{\beta/\alpha n},
\]

where

\[
D = T^{-1}(\frac{1}{2} \Omega_e) \cap \{ |x'| > \frac{1}{2} \varphi^{-1}(\frac{1}{2} \varepsilon), |x_n| < \frac{1}{2} \varphi(|x'|) \}.
\]

Verification of the inequality (\#). If \( d_e \leq 4 \varphi^{-1}(\varepsilon) = 4 \varepsilon^{1/\alpha} \), then by

\[
D \supset \{ \frac{1}{2} \varphi^{-1}(\frac{1}{2} \varepsilon) < |x'| < \frac{1}{2} \varphi^{-1}(\varepsilon), |x_n| < \frac{1}{16} \varepsilon \}
\]

\[
= \{ \frac{1}{2} (\frac{1}{2} \varepsilon)^{1/\alpha} < |x'| < \frac{1}{2} \varepsilon^{1/\alpha}, |x_n| < \frac{1}{16} \varepsilon \}
\]

we have \( \text{mes}(D) \geq C \varepsilon d_e^{-1} \). If \( d_e > 4 \varphi^{-1}(\varepsilon) \), then by the convexity of \( \Omega_e \) we have

\[
D \supset \{ \frac{1}{2} \varphi^{-1}(\varepsilon) < |x'| < \frac{1}{2} (\varphi^{-1}(\varepsilon) + d_e), |x_n| < \frac{1}{4} \varepsilon \}
\]

\[
= \{ \frac{1}{2} d_e < |x'| < \frac{1}{2} \varepsilon, |x_n| < \frac{1}{2} \varepsilon \},
\]

which also implies \( \text{mes}(D) \geq C \varepsilon d_e^{-1} \).

We therefore conclude

\[
u(0, \varepsilon) - u(0) = (\varepsilon d_e^{-1})^{2/n} M_e \geq (\varepsilon \cdot \varepsilon^{(n-1)/\alpha})^{2/n} \cdot C_3 \varepsilon^{\beta/\alpha n}
\]

\[
\geq C_3 \varepsilon^{(2\alpha + 2(n-1) + \beta)/n\alpha}.
\]

If

\[
(2\alpha + 2(n-1) + \beta)/n\alpha < 1,
\]

i.e., \( \beta < (n-2)\alpha - 2(n-1) \), then from the inequality (\#) we see that \( u(x) \) cannot be \( C^1 \) continuous. If

\[
(2\alpha + 2(n-1) + \beta)/n\alpha < 2,
\]

i.e., \( \beta < 2(n-1)(\alpha - 1) \), then the inequality (\#) shows that \( u(x) \) cannot be \( C^{1,1} \) smooth. This completes the proof.

**Remark 1.** In Example 3, since \( \alpha > 1 \), we have \( f(x) \in C^{\beta/\alpha}(\overline{\Omega}) \). Let \( \beta = 2(n-1)(\alpha - 1) - 1 \), and let \( \alpha > 1 \) be large enough. Then \( f \in C^{2(\alpha - 1) - \delta} \) for some \( \delta > 0 \) small, but the solution \( u \) is not \( C^{1,1} \) smooth. On the other hand, by the concavity of \( \det^{1/n}(D^2 u) \) and by the Alexandrov maximum principle one sees that if \( f^{1/n} \in C^{1,1}(\overline{\Omega}) \) and if \( \partial \Omega \) is \( C^2 \) smooth and strictly convex, then the solution \( u \) of the problem (1) is \( C^{1,1} \) smooth.

**Remark 2.** In [W] we proved the following \( C^2 \) regularity for the solution \( u \) of the problem (1).
Theorem 1. Let \( w(r) = \sup\{|f(x) - f(y)|; \ |x - y| \leq r\} \). If \( f \) is positive and \( \int_0^1 \frac{w(r)}{r} \, dr < \infty \), then \( u \in C^2(B_{1/2}) \).

Note that the function \( u(x, y) = \frac{x^2}{\log |\log r^2|} + \frac{y^2}{\log |\log r^2|} \), where \( r^2 = x^2 + y^2 \), is strictly convex and \( M(u) = 4 + \Theta(\log |\log r^2|/\log r^2) \) near the origin, but that \( u \notin W^{2, \infty} \). Hence, for continuous and positive \( f(x) \), the solution \( u \) of equation (1) may fail to lie in \( W^{2, \infty} \).

**References**


**Centre for Mathematical Sciences, Zhejiang University, Hangzhou, People’s Republic of China**